ACTA MATHEMATICA VIETNAMICA
Volume 36, Number 3, 2011, pp. 555–583

THE WRONSKIAN SOLUTIONS OF THE MODIFIED KORTEWEG-DE VRIES EQUATION

HA TIEN NGOAN AND NGUYEN HUY HOANG

Abstract. The generalized sufficient condition equations for the Wronskian determinants are proposed, which guarantee that they solve the modified Korteweg-de Vries equation (mKdV) in the bilinear form. Some new explicit Wronskian solutions to mKdV are constructed by applying variation of parameters.

1. Introduction

The modified Korteweg-de Vries (mKdV) equation is of the form

$$u_t + 6u^2 u_x + u_{xxx} = 0. \tag{1.1}$$

It is a soliton equation and has connections with shallow water wave equations. In equation (1.1), $u = u(x,t), (x,t) \in \mathbb{R}^2$ is the unknown function and subscripts with respect to x and t denote partial derivatives. There are some methods for solving the mKdV equation and other soliton equations, for example, the inverse scattering method [1, 12, 15], Hirota's method [1-3, 7-11, 13, 14, 16-18, 20], the Wronskian technique [2-11, 16-20], ...

In this paper we follow the Wronskian technique to construct explicit solutions of the mKdV equation (1.1). We note that in recent years, various classes of explicit solutions of many soliton equations were constructed in the form of Wronskian determinants [9-11, 19].

First, we mention some works that closely border on the problem to be discussed in this paper. Hirota and Satsuma [1, 3] have introduced the transformation

$$u(x,t) = i\Big(\ln \frac{\bar{f}(x,t)}{f(x,t)}\Big)_x, \tag{1.2}$$

where $i^2 = -1$, $\bar{f}(x,t)$ is the complex conjugate of $f(x,t)$, and $f(x,t) \neq 0$.

Received January 6, 2009; in revised form September 14, 2010.
2010 *Mathematics Subject Classification.* Primary 35Q53; Secondary 35Q51, 37K40.
Key words and phrases. Modified Korteweg-de Vries equation (mKdV), bilinear form of mKdV, Wronskian determinant, condition equations.
The first author is supported by NAFOSTED, grant code: 101.01.23.09.

They have proved that if $f(x,t)$ is a solution of the system of bilinear equations

$$(f_t\bar{f} - f\bar{f}_t) + (f_{xxx}\bar{f} - 3f_{xx}\bar{f}_x + 3f_x\bar{f}_{xx} - f\bar{f}_{xxx}) = 0, \tag{1.3}$$

$$f_{xx}\bar{f} - 2f_x\bar{f}_x + f\bar{f}_{xx} = 0, \tag{1.4}$$

then $u(x,t)$ defined by (1.2) is a solution of (1.1).

The above mentioned result of Hirota and Satsuma gives a way to construct a solution $u(x,t)$ of equation (1.1) from a solution $f(x,t)$ of system (1.3), (1.4). The converse problem of constructing a solution $f(x,t)$ from a solution $u(x,t)$ of (1.2) will be discussed in Corollary 2.2 in Section 2, which is a consequence of our main results.

Theorem 1.1. *Assume that $\alpha(x,t)$ and $\beta(x,t)$ are real-valued functions. Then a function of the form*

$$f(x,t) = e^{\alpha(x,t)+i\beta(x,t)} \tag{1.5}$$

satisfies system (1.3), (1.4) *if and only if α and β satisfy the system of equations*

$$\alpha_{xx} = 2\beta_x^2, \tag{1.6}$$

$$\beta_t + 8\beta_x^3 + \beta_{xxx} = 0. \tag{1.7}$$

Next, we discuss classes of N-soliton solutions of the bilinear system (1.3), (1.4). In [3], Hirota constructed an explicit N-soliton solution that is represented in the form

$$f_N(x,t) = \sum_{\mu=0,1} \exp\Big\{ \sum_{j=1}^{N} \mu_j\Big(\eta_j + i\frac{\pi}{2}\Big) + \sum_{1\leqslant l<j\leqslant N} \mu_l\mu_j A_{lj} \Big\}, \tag{1.8}$$

where $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$, the summation is taken over all possible combinations of $\mu_j = 0$ or 1 for $j = 1, 2, \ldots, N$, $\eta_j = 2k_jx - 8k_j^3t + \eta_j^{(0)}$, $\exp(A_{lj}) = \Big(\dfrac{k_j - k_l}{k_j + k_l}\Big)^2$ and $k_j, \eta_j^{(0)}$ are some real constants.

Nimmo and Freeman also constructed N-soliton solutions (1.8) by using the form of Wronskian determinant (see [4])

$$f(x,t) = \begin{vmatrix} \phi_1^{(0)} & \phi_1^{(1)} & \ldots & \phi_1^{(N-1)} \\ \phi_2^{(0)} & \phi_2^{(1)} & \ldots & \phi_2^{(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N^{(0)} & \phi_N^{(1)} & \ldots & \phi_N^{(N-1)} \end{vmatrix}, \tag{1.9}$$

where the entries $\phi_j^{(l)}$ of the Wronskian are $\phi_j^{(l)} = \phi_j^{(l)}(x,t) = \dfrac{\partial^l}{\partial x^l}\phi_j(x,t)$ with

$$\phi_j(x,t) = e^{i\frac{\pi}{4}}\Big[e^{\xi_j + i\frac{\pi}{4}} - (-1)^j e^{-\xi_j - i\frac{\pi}{4}}\Big], \tag{1.10}$$

where $\xi_j = k_jx - 4k_j^3t - \xi_j^{(0)}$, $\xi_j^{(0)}$ are some constants.

The function $\phi_j(x,t)$ in (1.10), which is used to determine the Wronskian determinant form of N-soliton solutions (1.8), was generalized by Nimmo, Freeman and Zhang (see [4,16]) in the following theorem.

Theorem 1.2 ([4,16])**.** *Suppose that for each* $j = 1, 2, \ldots, N$*,* $\phi_j(x,t)$ *satisfies the following system*

$$\phi_{jx} = -k_j\bar{\phi}_j, \tag{1.11}$$

$$\phi_{jt} = -4\phi_{jxxx}. \tag{1.12}$$

Then $f(x,t)$ *defined by* (1.9) *is a solution of* (1.3), (1.4).

The system (1.11), (1.12) is called the *condition equations*.

Our purpose in this paper is to construct some new classes of explicit solutions of the modified Korteweg-de Vries equation. In order to do this, we will extend the condition equations (1.11), (1.12) in such a way that $f(x,t)$ defined by (1.9) is still a solution of (1.3), (1.4). Our main result is the following theorem, in which we replace the condition equations (1.11), (1.12) by more general ones that guarantee the Wronskian determinant (1.9) to be a solution of the bilinear system (1.3), (1.4).

Theorem 1.3. *Assume that* $\phi_j = \phi_j(x,t)$ *satisfy the generalized condition equations*

$$\phi_{jx} = \sum_{l=1}^{N} a_{jl}(t)\bar{\phi}_l, \quad 1 \leqslant j \leqslant N, \tag{1.13}$$

$$\phi_{jt} = -4\phi_{jxxx} + \sum_{l=1}^{N} b_{jl}(t)\phi_l, \quad 1 \leqslant j \leqslant N, \tag{1.14}$$

where $A(t) \equiv (a_{jl}(t))_{N\times N}$ *and* $B(t) \equiv (b_{jl}(t))_{N\times N}$ *are real-valued continuously differentiable and continuous matrices, respectively, and satisfy the relation*

$$A_t + AB - BA = 0. \tag{1.15}$$

Then the Wronskian determinant $f(x,t)$ *defined by* (1.9) *satisfies the bilinear equations* (1.3), (1.4)*. Consequently, the function*

$$u(x,t) = i\Big(\ln\frac{\bar{f}}{f}\Big)_x \tag{1.16}$$

is a solution of the mKdV equation (1.1)*.*

The condition equations (1.13), (1.14) can be rewritten in the matrix form

$$\phi_x = A(t)\bar{\phi}, \tag{1.13$'$}$$

$$\phi_t = -4\phi_{xxx} + B(t)\phi, \tag{1.14$'$}$$

where $\phi(x,t) = (\phi_1(x,t), \phi_2(x,t), \ldots, \phi_N(x,t))^T$.
Our next result shows that the condition equations (1.13′), (1.14′) can be reduced to a simple canonical form.

Theorem 1.4. *The system* $(1.13'), (1.14')$ *can be transformed into the canonical form*

$$\psi_x = \Gamma\bar{\psi}, \tag{1.17}$$

$$\psi_t = -4\psi_{xxx}, \tag{1.18}$$

in which $B(t)$ *is the zero matrix and* $A(t)$ *is a constant matrix* Γ *in real Jordan form.*

Remark 1. Suppose that $F(x,t)$ and $G(x,t)$ are the real and imaginary parts, respectively, of the solution $f(x,t)$ of the system $(1.3), (1.4)$. Then the function $u(x,t)$ defined by (1.16) can be expressed in terms of $F(x,t)$ and $G(x,t)$ as follows [1]:

$$u(x,t) = 2\Big(\arctan\frac{G(x,t)}{F(x,t)}\Big)_x. \tag{1.19}$$

This paper is organized as follows. In Section 2 we present the proof of Theorem 1.1. The proof of Theorems 1.3 and 1.4 will be given in Section 3. In Section 4 the condition equations $(1.17), (1.18)$ will be discussed. In that section some explicit examples of Wronskian solutions containing many parameters will be constructed.

2. The relation between mKdV equation and its bilinear form

In this section we first give a proof of Theorem 1.1 and then derive two useful corollaries. Corollary 2.2 allows us to construct the solution $f(x,t)$ of the bilinear system $(1.3), (1.4)$ from a solution $u(x,t)$ of the mKdV equation (1.1).

Proof of Theorem 1.1. Substituting f of the form (1.5) into equation (1.4), we obtain

$$2f\bar{f}(\alpha_{xx} - 2\beta_x^2) = 0.$$

Thus, f of the form (1.5) is a solution of the bilinear equation (1.4) if and only if α and β satisfy equation (1.6). Next, by computing derivatives of f and $\bar{f}$ from (1.5), we get

$$f_x = f(\alpha_x + i\beta_x), \qquad \bar{f}_x = \bar{f}(\alpha_x - i\beta_x),$$
$$f_t = f(\alpha_t + i\beta_t), \qquad \bar{f}_t = \bar{f}(\alpha_t - i\beta_t),$$
$$f_{xx} = f(\alpha_{xx} + i\beta_{xx}) + f(\alpha_x + i\beta_x)^2,$$
$$\bar{f}_{xx} = \bar{f}(\alpha_{xx} - i\beta_{xx}) + \bar{f}(\alpha_x - i\beta_x)^2,$$
$$f_{xxx} = f(\alpha_{xxx} + i\beta_{xxx}) + 3f(\alpha_{xx} - i\beta_{xx})(\alpha_x + i\beta_x) + f(\alpha_x + i\beta_x)^3,$$
$$\bar{f}_{xxx} = \bar{f}(\alpha_{xxx} + i\beta_{xxx}) + 3\bar{f}(\alpha_{xx} - i\beta_{xx})(\alpha_x + i\beta_x) + \bar{f}(\alpha_x + i\beta_x)^3.$$

It follows that

$$(f_t\bar{f} - f\bar{f}_t) + (f_{xxx}\bar{f} - 3f_{xx}\bar{f}_x + 3f_x\bar{f}_{xx} - f\bar{f}_{xxx}) = 2if\bar{f}(\beta_t + \beta_{xxx} - 4\beta_x^3 + 6\alpha_{xx}\beta_x).$$

Combining the above equation with (1.3) we obtain

$$\beta_t + \beta_{xxx} - 4\beta_x^3 + 6\alpha_{xx}\beta_x = 0. \tag{2.1}$$

From (2.1) and using (1.6) we have (1.7). Hence, the proof of Theorem 1.1 is complete. □

It is easy to get from Theorem 1.1 the following corollaries.

Corollary 2.1. *Suppose that $f(x,t)$ is a solution of the form* (1.5) *of the system* (1.3), (1.4) *and that $h(x,t)$ is a function of the form*

$$h(x,t) = e^{h_1(t)x + h_2(t) + iC},$$

where $h_1(t)$ and $h_2(t)$ are arbitrary real-valued differentiable functions, and C is any real constant.
Then the product function $h(x,t)f(x,t)$ is also a solution of the system (1.3), (1.4).

Corollary 2.2. *Suppose that $u(x,t)$ is a solution of* (1.1) *and $f(x,t)$ is defined from $u(x,t)$ by the relation*

$$f(x,t) = e^{\alpha(x,t) + i\beta(x,t)},$$

where

$$\alpha(x,t) = \frac{1}{2}\int_0^x \int_0^\eta u^2(\xi,t)d\xi d\eta + C_1(t)x + C_2(t), \tag{2.2}$$

$$\beta(x,t) = \frac{1}{2}\int_0^x u(\xi,t)d\xi - \frac{1}{2}\int_0^t [2u^3(0,\tau) + u_{xx}(0,\tau)]d\tau + C, \tag{2.3}$$

$C_1(t), C_2(t)$ are real-valued differentiable functions, and C is a real constant. Then

1) *$f(x,t)$ is a solution of the system* (1.3), (1.4),
2) *$f(x,t)$ relates to $u(x,t)$ by* (1.2).

Proof. First, we will show that α, β satisfy (1.6), (1.7). Indeed, from (2.2) and (2.3) we have

$$\beta_x = \frac{1}{2}u, \tag{2.4}$$

$$\alpha_{xx} = \frac{1}{2}u^2, \tag{2.5}$$

which imply (1.6).

By using (2.3), (2.4) and (1.1) we have the relations

$$\begin{aligned}\beta_t(x,t) &= \frac{1}{2}\int_0^x u_t(\xi,t)d\xi - \frac{1}{2}[2u^3(0,t)+u_{xx}(0,t)]\\ &= \frac{1}{2}\int_0^x [-6u^2(\xi,t)u_\xi(\xi,t) - u_{\xi\xi\xi}(\xi,t)]d\xi - \frac{1}{2}[2u^3(0,t)+u_{xx}(0,t)]\\ &= -\frac{1}{2}\big[2u^3(x,t)+u_{xx}(x,t)\big]\\ &= -[8\beta_x^3(x,t)+\beta_{xxx}(x,t)].\end{aligned}$$

It follows that β satifies equation (1.7).

Therefore, α, β satisfy (1.6) and (1.7), and by virtue of Theorem 1.1 $f(x,t)$ satisfies the bilinear system (1.3), (1.4).
Next, since $f = e^{\alpha+i\beta}$ we have the relation

$$\frac{\bar f}{f} = e^{-2i\beta}.$$

Then by using (2.4) we obtain

$$u = 2\beta_x = i\Big(\ln\frac{\bar f}{f}\Big)_x.$$

Hence the proof of Corollary 2.2 is complete. □

3. Proof of Theorems 1.3 and 1.4

In what follows we need two properties of determinants that have been indicated in [3, 11].

Lemma 3.1 (Plücker relation)**.** *Let M be an $N\times(N-2)$ matrix, let O be an $N\times(N-2)$ zero matrix, and let a, b, c, d be N-column vectors. Then, the $2N\times 2N$ block determinant*

$$\begin{vmatrix} M & O & a & b & c & d\\ O & M & a & b & c & d\end{vmatrix}$$

is equal to zero. □

Lemma 3.2. *Let $|A|$, denoted by $|\alpha_1,\alpha_2,\dots,\alpha_N|$, be an $N\times N$ determinant with column vectors $\alpha_j = (\alpha_{1j},\alpha_{2j},\dots,\alpha_{Nj})^T$ and let $B=(b_{ij})$ be an $N\times N$ matrix. Then, the following formula holds:*

$$\sum_{j=1}^N |\alpha_1,\dots,\alpha_{j-1}, B\alpha_j, \alpha_{j+1},\dots,\alpha_N| = Tr(B)|A|. \tag{3.1}$$

□

Proof of Theorem 1.3. First, we show that the relation (1.15) is a necessary condition which guarantees that the bilinear system (1.13), (1.14) is consistent. Indeed, from (1.13′) and (1.14′), we have

$$\phi_{xt} = A_t\bar{\phi} - 4A\bar{\phi}_{xxx} + AB\bar{\phi},$$
$$\phi_{tx} = -4A\bar{\phi}_{xxx} + BA\bar{\phi},$$

and from $\phi_{xt} = \phi_{tx}$ (1.15) follows.
In the Wronskian determinant (1.9) we denote the single column

$$\phi^{(N-k)} = (\phi_1^{(N-k)}, \phi_2^{(N-k)}, \ldots, \phi_N^{(N-k)})^T,$$

by $N-k$ and the set of consecutive (ordered) columns $\phi^{(0)}, \phi^{(1)}, \ldots, \phi^{(N-k)}$ by $\widehat{N-k}$. Then, the Wronskian determinant f can be written in the compact form

$$f = W(\phi) = |\phi^{(0)}, \phi^{(1)}, \ldots, \phi^{(N-1)}| = |0, 1, \ldots, N-1| = |\widehat{N-1}|. \tag{3.2}$$

The derivatives of f can be easily computed in the compact form

$$f_x = |\widehat{N-2}, N|, \tag{3.3}$$

$$f_{xx} = |\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|, \tag{3.4}$$

$$f_{xxx} = |\widehat{N-4}, N-2, N-1, N| + 2|\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|. \tag{3.5}$$

Taking the complex conjugate in both sides of equation (1.13) we have

$$\bar{\phi}_{jx} = \sum_{l=1}^{N} a_{jl}(t)\phi_l, \quad j = 1, 2, \ldots, N.$$

Therefore

$$\bar{\phi}_j = \sum_{l=1}^{N} a_{jl}(t)\partial^{-1}\phi_l, \quad j = 1, 2, \ldots, N,$$

where $\partial^{-1}\phi_l$ is a primitive function of ϕ_l with respect to x.
Then, from the last equation, the relations

$$\bar{\phi}^{(0)} = A\partial^{-1}\phi^{(0)} = A\phi^{(-1)}, \quad \bar{\phi}^{(k)} = A\phi^{(k-1)},\ k \geq 1 \tag{3.6}$$

hold. Hence,

$$\begin{aligned}\bar{f} &= |\bar{\phi}^{(0)}, \bar{\phi}^{(1)}, \ldots, \bar{\phi}^{(N-1)}| = (\det A)|\phi^{(-1)}, \phi^{(0)}, \ldots, \phi^{(N-2)}| \\ &= (\det A)|-1, \widehat{N-2}|.\end{aligned} \tag{3.7}$$

From (3.7), we can compute the derivatives of $\bar{f}$ to be

$$\bar{f}_x = (\det A)|-1, \widehat{N-3}, N-1|, \tag{3.8}$$

$$\bar{f}_{xx} = (\det A)\big[|-1, \widehat{N-4}, N-2, N-1| + |-1, \widehat{N-3}, N|\big], \tag{3.9}$$

$$\begin{aligned}\bar{f}_{xxx} = (\det A)\big[&|-1, \widehat{N-5}, N-3, N-2, N-1| \\ &+ 2|-1, \widehat{N-4}, N-2, N| + |-1, \widehat{N-3}, N+1|\big].\end{aligned} \tag{3.10}$$

Next, we rewrite equation $(1.13')$ as follows

$$\phi_t^{(0)} = -4\phi^{(3)} + B\phi^{(0)}.$$

Then we obtain

$$\phi_t^{(j)} = -4\phi^{(j+3)} + B\phi^{(j)},\ j = 0, 1, \ldots, N-1.$$

Using the last equations and Lemma 3.2 we have

$$\begin{aligned} f_t &= \sum_{j=0}^{N-1} |\phi^{(0)}, \ldots, \phi^{(j-1)}, -4\phi^{(j+3)} + B\phi^{(j)}, \phi^{(j+1)}, \ldots, \phi^{(N-1)}| \\ &= -4\Big[|\widehat{N-4}, N-2, N-1, N| - |\widehat{N-3}, N-1, N+1| \\ &\quad + |\widehat{N-2}, N+2|\Big] + tr(B)|\widehat{N-1}|. \end{aligned} \tag{3.11}$$

From (3.6) and (3.11) we can also show that

$$\begin{aligned} \bar{f}_t = (\det A)\Big\{ &-4\Big[|-1, \widehat{N-5}, N-3, N-2, N-1| \\ &- |-1, \widehat{N-4}, N-2, N| + |-1, \widehat{N-3}, N+1|\Big] + tr(B)|-1, \widehat{N-2}|\Big\}. \end{aligned} \tag{3.12}$$

By using Lemma 3.2 we can obtain the following identities

$$tr(A^2)|\widehat{N-1}| = -|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|, \tag{3.13}$$

$$tr(A^2)|\widehat{N-2}, N| = -|\widehat{N-4}, N-2, N-1, N| + |\widehat{N-2}, N+2|, \tag{3.14}$$

$$tr(A^2)|-1, \widehat{N-2}| = -|-1, \widehat{N-4}, N-2, N-1| + |-1, \widehat{N-3}, N|, \tag{3.15}$$

$$\begin{aligned} tr(A^2)|-1, \widehat{N-3}, N-1| = -&|-1, \widehat{N-5}, N-3, N-2, N-1| \\ &+ |-1, \widehat{N-3}, N+1|. \end{aligned} \tag{3.16}$$

Substituting the derivatives of f and $\bar{f}$ into the left-hand side of equations (1.3) we have

$$
\begin{aligned}
(3.17)\quad \Delta &\equiv f_t\bar{f} - f\bar{f}_t + f_{xxx}\bar{f} - 3f_{xx}\bar{f}_x + 3f_x\bar{f}_{xx} - f\bar{f}_{xxx} \\
&= -3(\det A)\Big[|\widehat{N-4}, N-2, N-1, N| - 2|\widehat{N-3}, N-1, N+1| \\
&\qquad + |\widehat{N-2}, N+2|\Big]|-1, \widehat{N-2}| \\
&+ 3(\det A)\Big[3|-1, \widehat{N-5}, N-3, N-2, N-1| \\
&\quad - 6|-1, \widehat{N-4}, N-2, N| + 3|-1, \widehat{N-3}, N+1|\Big]|\widehat{N-1}| \\
&+ 3(\det A)\Big[|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|\Big]|-1, \widehat{N-3}, N-1| \\
&- 3(\det A)\Big[|-1, \widehat{N-4}, N-2, N-1| + |-1, \widehat{N-3}, N|\Big]|\widehat{N-2}, N|.
\end{aligned}
$$

By the identities (3.12)-(3.16) we may reduce Δ in (3.17) to the form

$$
\begin{aligned}
\Delta = (\det A)\Big[&6|-1, \widehat{N-3}, N+1||\widehat{N-1}| + 6|\widehat{N-3}, N-1, N+1||-1, \widehat{N-2}| \\
&- 6|-1, \widehat{N-3}, N-1||\widehat{N-2}, N+1|\Big] \\
+ (\det A)\Big[&6|-1, \widehat{N-4}, N-2, N||\widehat{N-1}| + 6|\widehat{N-4}, N-2, N-1, N||-1, \widehat{N-2}| \\
&- 6|-1, \widehat{N-4}, N-2, N-1||\widehat{N-2}, N|\Big].
\end{aligned}
$$

By using the Laplace expansion by $N \times N$ minors we can rewrite Δ in the form

(3.18)

$$
\begin{aligned}
\Delta &= (-1)^{N-2}(3\det A)\begin{vmatrix} \widehat{N-3} & O & -1 & N-2 & N-1 & N+1 \\ O & \widehat{N-3} & -1 & N-2 & N-1 & N+1 \end{vmatrix} \\
&+(-1)^{N-2}(3\det A)\begin{vmatrix} \widehat{N-4}, N-2 & O & -1 & N-3 & N-1 & N \\ O & \widehat{N-4}, N-2 & -1 & N-3 & N-1 & N \end{vmatrix}.
\end{aligned}
$$

By virtue of Lemma 3.1, the two determinants on the right-hand side of (3.18) are zero. Therefore f satisfies the bilinear equation (1.3).

In a similar way, from the left-hand side of equation (1.4) we get

$$
\begin{aligned}
(3.19)\quad f_{xx}\bar{f} - 2f_x\bar{f}_x + f\bar{f}_{xx} &= (\det A)\Big[2|-1, \widehat{N-2}||\widehat{N-3}, N-1, N| \\
&- 2|\widehat{N-2}, N||-1, \widehat{N-3}, N-1| + 2|-1, \widehat{N-3}, N||\widehat{N-1}|\Big] \\
&= (-1)^{N-2}(\det A)\begin{vmatrix} \widehat{N-3} & O & -1 & N-2 & N-1 & N \\ O & \widehat{N-3} & -1 & N-2 & N-1 & N \end{vmatrix} = 0.
\end{aligned}
$$

Thus, f is a Wronskian solution of the bilinear equations (1.3), (1.4) and $u = i\Big(\ln\dfrac{\bar f}{f}\Big)_x$ is a solution of the mKdV equation (1.1). The proof of Theorem 1.3 is complete. □

Now we bring out some overview for the condition equations (1.13) and (1.14).

Lemma 3.3 ([19])**.** *Suppose that $B = (b_{jl}(t))_{N\times N} \in C[a,b]$ is a real-valued matrix depending continuously on t. Then there exists a non-singular continuously differentiable real $N \times N$ matrix $H(t)$ satisfying the equation*

$$H_t(t) = B(t)H(t), \tag{3.20}$$

$$H(a) = D, \tag{3.21}$$

where D is an invertible constant matrix. □

By using the matrix $H(t)$ in Lemma 3.3 and putting

$$\phi = H(t)\psi \tag{3.22}$$

it holds $\phi^{(j)} = H(t)\psi^{(j)}$ and

$$\begin{aligned} W(\phi) &= |H(t)\psi^{(0)}, H(t)\psi^{(1)}, \ldots, H(t)\psi^{(N-1)}| \\ &= \det(H(t))|\psi^{(0)}, \psi^{(1)}, \ldots, \psi^{(N-1)}| = (\det H(t))W(\psi). \end{aligned}$$

We also obtain

$$\frac{\partial}{\partial x}\ln\Big(\frac{\overline{W}(\phi)}{W(\phi)}\Big) = \frac{\partial}{\partial x}\ln\Big(\frac{\overline{W}(\psi)}{W(\psi)}\Big).$$

From the above equation and (1.2) it follows that the vector functions ϕ and ψ give the same solutions to the mKdV equation. Thus, we can replace the Wronskian $W(\phi)$ by $W(\psi)$.

Lemma 3.4. *The initial matrix D of problem* (3.20), (3.21) *can be chosen in such a way that $\psi(x,t)$ as determined by* (3.22) *is a solution of the condition equations*

$$\psi_x = \Gamma\bar\psi, \tag{3.23}$$

$$\psi_t = -4\psi_{xxx}, \tag{3.24}$$

where Γ is the constant matrix in the real Jordan form of the matrix $A(a)$.

Proof. Since $\phi = H(t)\psi$ we have $\bar\phi = H(t)\bar\psi$ and $\psi = H^{-1}\phi$. By using (1.13) we obtain

$$\psi_x = H^{-1}\phi_x = H^{-1}A\bar\phi = (H^{-1}AH)\bar\psi.$$

Next, from Lemma 3.3 we have

$$\phi_t = H_t\psi + H\psi_t = BH\psi + H\psi_t,$$

and by using (1.14) we obtain

$$\begin{aligned} \psi_t &= H^{-1}\big[\phi_t - BH\psi\big] \\ &= H^{-1}\big[-4\phi_{xxx} + B\phi - BH\psi\big] \\ &= -4\psi_{xxx}. \end{aligned}$$

Next, note that $H_t^{-1} = -H^{-1}H_tH^{-1}$, so by using (1.15) and (3.20) we have

$$\begin{aligned}(H^{-1}AH)_t &= H_t^{-1}AH + H^{-1}A_tH + H^{-1}AH_t \\ &= H^{-1}(A_t + AB - BA)H = 0.\end{aligned}$$

Then $H^{-1}AH$ must be a constant matrix, and therefore, $H^{-1}(t)A(t)H(t) = D^{-1}A(a)D$. We can choose the initial matrix D in such way that $\Gamma = D^{-1}A(a)D$ is a matrix in the real Jordan form of the matrix $A(a)$, Lemma 3.4 is proved. □

Proof of Theorem 1.4. In (3.20) we can choose the matrix D such that $\Gamma = D^{-1}A(a)D$ is an $N \times N$ matrix in the real Jordan form. By virtue of Lemmas 3.3 and 3.4, the condition equations $(1.13'), (1.14')$ can be transformed into the system (3.23), (3.24). Thus, the system (1.13), (1.14) can be reduced to the system (1.17), (1.18), that is

$$\begin{aligned}\psi_x &= \Gamma\bar{\psi}, \\ \psi_t &= -4\psi_{xxx},\end{aligned}$$

where Γ is a constant $N \times N$ matrix in the real canonical Jordan form. □

4. Solutions of the condition equations

In this section we will discuss the condition equations of the form (1.17), (1.18). Since Γ is a real canonical Jordan matrix, we can separate the condition equations into several independent systems of the form

$$\phi_x = \Gamma_m\bar{\phi}, \tag{4.1}$$

$$\phi_t = -4\phi_{xxx}, \tag{4.2}$$

where Γ_m is a real Jordan block of order m.
We consider various cases of real Jordan blocks Γ_m and give general solutions of (4.1), (4.2) in each case. We show that the set of all solutions of the system (4.1), (4.2) form a vector space over $\mathbb{R}$ of dimension $2m$. Then, some classes of explicit solutions of the mKdV equation (1.1) will be obtained below.

4.1. **Case: The Jordan block Γ_m is diagonal.** First, we consider the case that $\Gamma_m = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m),\ \lambda_j \in \mathbb{R},\ j = 1, 2, \ldots, m$. For each index j we can solve the corresponding equations to obtain the general solutions

$$\phi_j = C_{j1}e^{(\lambda_j x - 4\lambda_j^3 t)} + iC_{j2}e^{-(\lambda_j x - 4\lambda_j^3 t)}, \tag{4.3}$$

where C_{j1} and C_{j2} are arbitrary real constants.
This means that in this case the vector space over $\mathbb{R}$ of all solutions $\phi(x,t)$ of (4.1), (4.2) is of dimension $2m$.
If we put $g_j = e^{(\lambda_j x - 4\lambda_j^3 t)}$, then (4.3) can be written as

$$\phi_j = \sum_{\epsilon_j = \pm 1} \Big[\frac{1+\epsilon_j}{2}C_{j1} + i\frac{1-\epsilon_j}{2}C_{j2}\Big] g_j^{\epsilon_j}. \tag{4.4}$$

From (1.9), we have for $N = m$

$$\begin{aligned} f_m :=& W(\phi_1, \phi_2, \dots, \phi_m) \\ =& \sum_{(\epsilon_1, \epsilon_2, \dots, \epsilon_m)} \Big\{ \prod_{j=1}^{m} \Big[\frac{1+\epsilon_j}{2} C_{j1} + i \frac{1-\epsilon_j}{2} C_{j2} \Big] W(g_1^{\epsilon_1}, g_2^{\epsilon_2}, \dots, g_m^{\epsilon_m}) \Big\}, \end{aligned} \tag{4.5}$$

where the summation is taken over all m-tuples $(\epsilon_1, \epsilon_2, \dots, \epsilon_m)$ with $\epsilon_j = \pm 1$ for $j = 1, 2, \dots, m$.

By computing the x-direction derivatives of g_j we obtain

$$g_j^{(k)} := \frac{\partial^k}{\partial x^k} g_j = \lambda_j^k g_j, \quad k = 1, 2, \dots \tag{4.6}$$

$$\big(g_j^{\epsilon_j}\big)^{(k)} = \epsilon_j^k \lambda_j^k g_j^{\epsilon_j}, \quad k = 1, 2, \dots \tag{4.7}$$

From (4.6) and (4.7) we have the relation

$$\begin{aligned} W(g_1^{\epsilon_1}, g_2^{\epsilon_2}, \dots, g_m^{\epsilon_m}) &= \prod_{j=1}^{m} g_j^{\epsilon_j} \begin{vmatrix} 1 & \epsilon_1\lambda_1 & \dots & \epsilon_1^{m-1}\lambda_1^{m-1} \\ 1 & \epsilon_2\lambda_2 & \dots & \epsilon_2^{m-1}\lambda_2^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \epsilon_m\lambda_m & \dots & \epsilon_m^{m-1}\lambda_m^{m-1} \end{vmatrix} \\ &= \prod_{j=1}^{m} g_j^{\epsilon_j} \prod_{1 \leqslant k < l \leqslant m} (\epsilon_l \lambda_l - \epsilon_k \lambda_k). \end{aligned} \tag{4.8}$$

Next, by using (4.5) and (4.8) we have

$$f_m = \sum_{(\epsilon_1, \epsilon_2, \dots, \epsilon_m)} \Big\{ \Big(\prod_{j=1}^{m} \Big[\frac{1+\epsilon_j}{2} C_{j1} + i \frac{1-\epsilon_j}{2} C_{j2} \Big] g_j^{\epsilon_j} \Big) \prod_{1 \leqslant k < l \leqslant m} (\epsilon_l \lambda_l - \epsilon_k \lambda_k) \Big\}. \tag{4.9}$$

In the representation (4.9) the summation is taken over 2^m terms which correspond to the set of m-tuples $(\epsilon_1, \epsilon_2, \dots, \epsilon_m)$ with $\epsilon_j = \pm 1$. The set of Wronskian solutions $f_m(x,t)$ represented by (4.9) contains $2m$ arbitrary real parameters $C_{j1}, C_{j2},\ j = 1, 2, \dots, m$.

We note that the factor

$$\Big[\frac{1+\epsilon_j}{2} C_{j1} + i \frac{1-\epsilon_j}{2} C_{j2} \Big]$$

is either real-valued or pure-imaginary-valued. It is pure-imaginary-valued if and only if $\epsilon_j = -1$. So, in case the m-tuple $(\epsilon_1, \epsilon_2, \dots, \epsilon_m)$ has an even number of components $\epsilon_j = -1$ then the corresponding term is real-valued. Then the set of those m-tuples is denoted by X_1. In other words, X_1 is the set of all m-tuples $(\epsilon_1, \epsilon_2, \dots, \epsilon_m)$ such that $\prod_{j=1}^{m} \epsilon_j = 1$. The set of all other m-tuples is denoted by X_2, i.e. X_2 is the set of all m-tuples $(\epsilon_1, \epsilon_2, \dots, \epsilon_m)$ such that $\prod_{j=1}^{m} \epsilon_j = -1$.

We can represent the solution $f_m(x,t)$ in the form

$$f_m(x,t) = F_m(x,t) + iG_m(x,t),$$

where $F_m(x,t)$ and $G_m(x,t)$ are its real and imaginary parts, respectively. Namely,

(4.10)
$$F_m = \sum_{(\epsilon_1,\epsilon_2,\ldots,\epsilon_m)\in X_1} \Big\{\Big(\prod_{j=1}^{m}\Big[\frac{1+\epsilon_j}{2}C_{j1}+i\frac{1-\epsilon_j}{2}C_{j2}\Big]g_j^{\epsilon_j}\Big)\prod_{1\leqslant k<l\leqslant m}(\epsilon_l\lambda_l-\epsilon_k\lambda_k)\Big\},$$

(4.11)
$$G_m = (-i)\sum_{(\epsilon_1,\epsilon_2,\ldots,\epsilon_m)\in X_2} \Big\{\Big(\prod_{j=1}^{m}\Big[\frac{1+\epsilon_j}{2}C_{j1}+i\frac{1-\epsilon_j}{2}C_{j2}\Big]g_j^{\epsilon_j}\Big)\prod_{1\leqslant k<l\leqslant m}(\epsilon_l\lambda_l-\epsilon_k\lambda_k)\Big\}.$$

According to (1.19), the functions

$$u(x,t) = 2\Big(\arctan\frac{G_m}{F_m}\Big)_x \tag{4.12}$$

are solutions of (1.1). Note that $u(x,t)$ in (4.12) contains $2m$ arbitrary real parameters.

To connect our results with well-known ones we will write the functions ϕ_j in (4.3) in another form. Suppose that $C_{j1}C_{j2}\neq 0$ for all $j=1,2,\ldots,m$. Then, we can rewrite ϕ_j as follows:

$$\phi_j = C_{j2}\sqrt{\Big|\frac{C_{j1}}{C_{j2}}\Big|}e^{i\frac{\pi}{4}}\Big[e^{\xi_j+i\frac{\pi}{4}}-(-1)^{\alpha_j}e^{-\xi_j-i\frac{\pi}{4}}\Big], \tag{4.13}$$

where $\frac{|C_{j1}|}{\sqrt{|C_{j1}C_{j2}|}} = e^{\xi_j^{(0)}}$, $\xi_j = -\lambda_j x + 4\lambda_j^3 t - \xi_j^{(0)}$ and $\alpha_j = \frac{1}{2}\Big(1+\frac{C_{j1}C_{j2}}{|C_{j1}C_{j2}|}\Big)$.
If $m=N=2$ and if we choose

$$\phi_1 = ie^{\xi_1}+e^{-\xi_1}, \quad \phi_2 = ie^{\xi_2}-e^{-\xi_2},$$

then

$$\begin{aligned} f &= \begin{vmatrix} ie^{\xi_1}+e^{-\xi_1} & \lambda_1(-ie^{\xi_1}+e^{-\xi_1}) \\ ie^{\xi_2}-e^{-\xi_2} & \lambda_2(-ie^{\xi_2}-e^{-\xi_2}) \end{vmatrix} \\ &= (\lambda_2-\lambda_1)(e^{\xi_1+\xi_2}-e^{-(\xi_1+\xi_2)}) - i(\lambda_1+\lambda_2)(e^{\xi_1-\xi_2}+e^{-(\xi_1-\xi_2)}). \end{aligned}$$

Thereby, we can obtain the 2-soliton solution

$$\begin{aligned} u = i\Big(\ln\frac{\bar f}{f}\Big)_x &= 2\Big\{\arctan\Big[\Big(\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}\Big)\frac{e^{\xi_1-\xi_2}+e^{-(\xi_1-\xi_2)}}{e^{\xi_1+\xi_2}-e^{-(\xi_1+\xi_2)}}\Big]\Big\}_x \\ &= 2\Big\{\arctan\Big[\frac{(\lambda_1+\lambda_2)\cosh(\xi_1-\xi_2)}{(\lambda_1-\lambda_2)\sinh(\xi_1+\xi_2)}\Big]\Big\}_x. \end{aligned} \tag{4.14}$$

On the other hand, if we choose

$$\phi_1 = ie^{\xi_1}+e^{-\xi_1}, \quad \phi_2 = ie^{\xi_2}+e^{-\xi_2},$$

then

$$u = 2\Big\{\arctan\Big[\frac{(\lambda_1+\lambda_2)\sinh(\xi_2-\xi_1)}{(\lambda_1-\lambda_2)\cosh(\xi_1+\xi_2)}\Big]\Big\}_x. \tag{4.15}$$

We will show that if the set of real numbers λ_j is fixed, then our family of solutions f_m (4.9) contains the set of solutions represented by (1.8) as a particular case. Indeed, if the functions ϕ_j in (4.13) are chosen as

$$\phi_j = e^{i\frac{\pi}{4}}\Big[e^{\xi_j + i\frac{\pi}{4}} - (-1)^j e^{-\xi_j - i\frac{\pi}{4}}\Big], \tag{4.16}$$

where $\xi_j = k_j x - 4k_j^3 t - \xi_j^{(0)},\ k_j = -\lambda_j,\ j = 1,2,\dots,N,\quad (m = N)$, then the Wronskian solution f_N can be described by the following corollary.

Corollary 4.1 ([17,18])**.** *If $\phi_j, j = 1,2,\dots,N$ are chosen as in* (4.16), *then the Wronskian solution f_N can be written in the form*

(4.17)

$$f_N = \Big(\prod_{j=1}^{N} e^{-\xi_j}\Big)\Big(\prod_{1\leqslant l<j\leqslant N}(k_j - k_l)\Big)\sum_{\mu=0,1}\exp\Big\{\sum_{j=1}^{N}\mu_j\Big(\eta_j + i\frac{\pi}{2}\Big) + \sum_{1\leqslant l<j\leqslant N}\mu_l\mu_j A_{lj}\Big\},$$

where $\eta_j = 2\xi_j - \dfrac{1}{2}\sum_{l=1, l\neq j}^{N} A_{lj},\ \exp(A_{lj}) = \Big(\dfrac{k_j - k_l}{k_j + k_l}\Big)^2,$

which is connected to the solution (1.8) *by Corollary 2.1.* □

4.2. **Case: Γ_m is a Jordan block with a real eigenvalue λ.** In this subsection we consider the system (4.1), (4.2) in case the matrix Γ_m is of the form

$$\Gamma_m = \begin{pmatrix} \lambda & & & 0 \\ 1 & \lambda & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda \end{pmatrix}_{m\times m}, \tag{4.18}$$

where λ is a real eigenvalue.
In this case, by using the transformation

$$\begin{aligned} \phi_1 &= \tilde{\phi}_1, \\ \phi_2 &= \tilde{\phi}_2 + \frac{\partial}{\partial\lambda}\tilde{\phi}_1, \\ &\dots \\ \phi_m &= \tilde{\phi}_m + \frac{\partial}{\partial\lambda}\tilde{\phi}_{m-1} + \dots + \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial\lambda^{m-1}}\tilde{\phi}_1, \end{aligned}$$

we can obtain that

$$\tilde{\phi}_{jx} = \lambda\bar{\tilde{\phi}}_j, \tag{4.19}$$

$$\tilde{\phi}_{jt} = -4\tilde{\phi}_{jxxx}, \tag{4.20}$$

for $j = 1,2,\dots,m$.

As shown in subsection 4.1 it follows from (4.19) and (4.20) that

$$\tilde{\phi}_j = C_{j1}\,e^{\lambda x - 4\lambda^3 t} + iC_{j2}\,e^{-\lambda x + 4\lambda^3 t},\ j = 1,2,\dots,m,$$

where $C_{11}, C_{21},\dots,C_{m1};\ C_{12}, C_{22},\dots,C_{m2}$ are $2m$ arbitrary real constants.

Thus, we can obtain the following theorem.

Theorem 4.1. *The general solutions* ϕ *of system* (4.1), (4.2) *with the Jordan block* Γ_m *in* (4.18) *are of the form*

(4.21)

$$\begin{pmatrix}\phi_1\\ \phi_2\\ \vdots\\ \phi_m\end{pmatrix}=\begin{pmatrix}1 & 0 & \dots & 0\\ \frac{\partial}{\partial\lambda} & 1 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial\lambda^{m-1}} & \frac{1}{(m-2)!}\frac{\partial^{m-2}}{\partial\lambda^{m-2}} & \dots & 1\end{pmatrix}\begin{pmatrix}C_{11}e^{\lambda x-4\lambda^3 t}+iC_{12}e^{-\lambda x+4\lambda^3 t}\\ C_{21}e^{\lambda x-4\lambda^3 t}+iC_{22}e^{-\lambda x+4\lambda^3 t}\\ \vdots\\ C_{m1}e^{\lambda x-4\lambda^3 t}+iC_{m2}e^{-\lambda x+4\lambda^3 t}\end{pmatrix},$$

where $C_{11}, C_{21}, \dots, C_{m1},\ C_{12}, C_{22}, \dots, C_{m2}$ *are arbitrary real constants.* □

By (4.21) we can write ϕ_j as

$$\phi_j = P_{j1}g + iP_{j2}h, \qquad j = 1, 2, \dots, m, \tag{4.22}$$

where $g = e^{\lambda x-4\lambda^3 t}$, $h = g^{-1} = e^{-\lambda x+4\lambda^3 t}$ and $P_{j1} \equiv P_{j1}(x, t), P_{j2} \equiv P_{j2}(x, t)$ are some polynomials with respect to variables x and t determined by the relations

$$P_{j1}(x,t) = e^{-\lambda x+4\lambda^3 t}\sum_{l=1}^{j} C_{l1}\frac{1}{(j-l)!}\frac{\partial^{j-l}}{\partial\lambda^{j-l}}e^{\lambda x-4\lambda^3 t}, \tag{4.23}$$

$$P_{j2}(x,t) = e^{\lambda x-4\lambda^3 t}\sum_{l=1}^{j} C_{l2}\frac{1}{(j-l)!}\frac{\partial^{j-l}}{\partial\lambda^{j-l}}e^{-\lambda x+4\lambda^3 t}. \tag{4.24}$$

Next, the derivatives of ϕ_j can be written as

$$\phi_j^{(k)} = \frac{\partial^k}{\partial x^k}\phi_j = Q_{j1}(k)g + iQ_{j2}(k)h, \qquad j = 1, 2, \dots, m, \tag{4.25}$$

where the polynomials $Q_{j1}(k), Q_{j2}(k)$ are calculated by the formulas

$$Q_{j1}(k) \equiv Q_{j1}(x,t,k) = \sum_{l=0}^{k} C_k^l\lambda^{k-l}\frac{\partial^l}{\partial x^l}P_{j1}(x,t), \tag{4.26}$$

$$Q_{j2}(k) \equiv Q_{j2}(x,t,k) = \sum_{l=0}^{k} C_k^l(-1)^{k-l}\lambda^{k-l}\frac{\partial^l}{\partial x^l}P_{j2}(x,t). \tag{4.27}$$

We now can rewrite (4.25) as

$$\phi_j^{(k)} = \sum_{\epsilon_j=\pm1}\big[\frac{1+\epsilon_j}{2}Q_{j1}(k) + i\frac{1-\epsilon_j}{2}Q_{j2}(k)\big]g^{\epsilon_j}. \tag{4.28}$$

We set

$$D(\epsilon_1, \epsilon_2, \dots, \epsilon_m) = \det\big[\frac{1+\epsilon_j}{2}Q_{j1}(k) + i\frac{1-\epsilon_j}{2}Q_{j2}(k)\big]_{1\leqslant j,k\leqslant m}. \tag{4.29}$$

Note that in the above determinant, if $\epsilon_j = 1$ then the j-th row is a real-valued vector, and if $\epsilon_j = -1$ then the j-th row is a pure-imaginary-valued vector. Now we define the sets X_1, X_2 as in subsection 4.1. Then, the value of determinant (4.29) is real if $(\epsilon_1, \epsilon_2, \dots, \epsilon_m) \in X_1$, and is pure-imaginary if $(\epsilon_1, \epsilon_2, \dots, \epsilon_m) \in X_2$.

By using (4.28) and (4.29) we can transform the Wronskian determinant $f_m = W(\phi_1, \phi_2, \ldots, \phi_m)$ into the form

$$f_m = \sum_{(\epsilon_1,\epsilon_2,\ldots,\epsilon_m)} \Big(\prod_{j=1}^{m} g^{\epsilon_j}\Big) D(\epsilon_1, \epsilon_2, \ldots, \epsilon_m). \tag{4.30}$$

By using the sets X_1, X_2 we separate f_m into real and imaginary parts. Note that $(\epsilon_1, \epsilon_2, \ldots, \epsilon_m) \in X_1$ if and only if $\sum_{j=1}^{m} \epsilon_j = m - 4\alpha$, where α is an integer such that $0 \leqslant \alpha \leqslant m/2$. On the other hand, $(\epsilon_1, \epsilon_2, \ldots, \epsilon_m) \in X_2$ if and only if $\sum_{j=1}^{m} \epsilon_j = m - 4\alpha - 2$, where α is an integer such that $0 \leqslant \alpha < m/2$. So, we have the following representation of f_m:

$$f_m = F_m + iG_m,$$
$$u = 2\Big(\arctan \frac{G_m}{F_m}\Big)_x,$$

where

$$F_m = \sum_{0\leqslant\alpha\leqslant m/2} g^{m-4\alpha}\Big[\sum_{\epsilon_1+\epsilon_2+\ldots+\epsilon_m=m-4\alpha} D(\epsilon_1, \epsilon_2, \ldots, \epsilon_m)\Big], \tag{4.31}$$

$$G_m = (-i) \sum_{0\leqslant\alpha<m/2} g^{m-4\alpha-2}\Big[\sum_{\epsilon_1+\epsilon_2+\ldots+\epsilon_m=m-4\alpha-2} D(\epsilon_1, \epsilon_2, \ldots, \epsilon_m)\Big]. \tag{4.32}$$

As shown in subsection 4.1, in this case the Wronskian solutions f_m depend on $2m$ arbitrary real constants.

Next, we give some examples of the so-called bi-directional Wronskian. To do this, we choose $C_{21} = C_{22} = \ldots = C_{m1} = C_{m2} = 0$ in (4.21). Then we have the following Wronskian:

$$f_m = W\Big(\phi_1, \frac{\partial\phi_1}{\partial\lambda}, \frac{1}{2!}\frac{\partial^2\phi_1}{\partial\lambda^2}, \ldots, \frac{1}{(m-1)!}\frac{\partial^{m-1}\phi_1}{\partial\lambda^{m-1}}\Big). \tag{4.33}$$

We note that the function f_m in (4.33) can be regarded as a Wronskian determinant for both x-direction and λ-direction. This is why it is called bi-directional.

We have from (4.33) the following bi-directional Wronskian solution for the case $m = 2$:

$$f = (C_{11}^2 e^{2\lambda x - 8\lambda^3 t} + C_{12}^2 e^{-2\lambda x + 8\lambda^3 t}) + 4\lambda C_{11} C_{12}(x - 12\lambda^2 t)i. \tag{4.34}$$

Due to (1.19) the solution u of the mKdV equation for $m = 2$ is given by

$$u = 2\Big(\arctan \frac{4\lambda C_{11}C_{12}(x - 12\lambda^2 t)}{C_{11}^2 e^{2\lambda x - 8\lambda^3 t} + C_{12}^2 e^{-2\lambda x + 8\lambda^3 t}}\Big)_x. \tag{4.35}$$

It follows from (4.33) that the bi-directional Wronskian solution for the case $m = 3$ is

(4.36)

$$\begin{aligned} f = {} & C_{11}e^{-\lambda x + 4\lambda^3 t}\big\{C_{11}^2 e^{4\lambda x - 16\lambda^3 t} + C_{12}^2\big[4\lambda(x - 36\lambda^2 t) + 1 + 8\lambda^2(x - 12\lambda^2 t)^2\big]\big\} \\ & + C_{12}e^{\lambda x - 4\lambda^3 t}\big\{C_{12}^2 e^{-4\lambda x + 16\lambda^3 t} + C_{11}^2\big[-4\lambda(x - 36\lambda^2 t) + 1 + 8\lambda^2(x - 12\lambda^2 t)^2\big]\big\}i. \end{aligned}$$

Analogously, the solution u of the mKdV equation for $m = 3$ is

(4.37)

$$u(x,t) = 2\left(\arctan \frac{C_{12}e^{\lambda x-4\lambda^3 t}\{C_{12}^2 e^{-4\lambda x+16\lambda^3 t}+C_{11}^2\left[-4\lambda(x-36\lambda^2 t)+1+8\lambda^2(x-12\lambda^2 t)^2\right]\}}{C_{11}e^{-\lambda x+4\lambda^3 t}\{C_{11}^2 e^{4\lambda x-16\lambda^3 t}+C_{12}^2\left[4\lambda(x-36\lambda^2 t)+1+8\lambda^2(x-12\lambda^2 t)^2\right]\}}\right)_x.$$

4.3. **Case: Γ_m is a 2×2 real Jordan block with a pair of complex eigenvalues.** We consider the case that the matrix $\Gamma_m = \Gamma_2$ corresponds to a pair of complex eigenvalues $\lambda = \alpha + i\beta$ and $\overline{\lambda} = \alpha - i\beta$, i.e.

$$\Gamma_2 = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}, \tag{4.38}$$

where α, β are real constants, $\beta \neq 0$.
In this case, system (4.1), (4.2) is of the form

$$\phi_{1x} = \alpha\bar{\phi}_1 - \beta\bar{\phi}_2, \tag{4.39}$$
$$\phi_{2x} = \beta\bar{\phi}_1 + \alpha\bar{\phi}_2, \tag{4.40}$$
$$\phi_{1t} = -4\phi_{1xxx}, \tag{4.41}$$
$$\phi_{2t} = -4\phi_{2xxx}. \tag{4.42}$$

Theorem 4.2. *The general solutions ϕ of* (4.39)-(4.42) *can be given by*

(4.43)

$$\phi_1 = e^{\alpha x-4\alpha(\alpha^2-3\beta^2)t}\{C_1\cos[\beta x-4\beta(3\alpha^2-\beta^2)t]-C_2\sin[\beta x-4\beta(3\alpha^2-\beta^2)t]\}$$
$$+ie^{-\alpha x+4\alpha(\alpha^2-3\beta^2)t}\{C_3\cos[\beta x-4\beta(3\alpha^2-\beta^2)t]+C_4\sin[\beta x-4\beta(3\alpha^2-\beta^2)t]\},$$

(4.44)

$$\phi_2 = e^{\alpha x-4\alpha(\alpha^2-3\beta^2)t}\{C_1\sin[\beta x-4\beta(3\alpha^2-\beta^2)t]+C_2\cos[\beta x-4\beta(3\alpha^2-\beta^2)t]\}$$
$$+ie^{-\alpha x+4\alpha(\alpha^2-3\beta^2)t}\{-C_3\sin[\beta x-4\beta(3\alpha^2-\beta^2)t]+C_4\cos[\beta x-4\beta(3\alpha^2-\beta^2)t]\},$$

where C_1, C_2, C_3, C_4 are arbitary real constants.

Proof. Putting $\phi_1 = u_1 + iv_1,\ \phi_2 = u_2 + iv_2$, we can easily show that

$$u_{1x} = \alpha u_1 - \beta u_2, \tag{4.45}$$
$$u_{2x} = \beta u_1 + \alpha u_2, \tag{4.46}$$
$$u_{1t} = -4u_{1xxx}, \tag{4.47}$$
$$u_{2t} = -4u_{2xxx}, \tag{4.48}$$

and

$$v_{1x} = -\alpha v_1 + \beta v_2, \tag{4.49}$$

$$v_{2x} = -\beta v_1 - \alpha v_2, \tag{4.50}$$

$$v_{1t} = -4v_{1xxx}, \tag{4.51}$$

$$v_{2t} = -4v_{2xxx}. \tag{4.52}$$

First, we solve system (4.45)-(4.48) with respect to u_1, u_2. By eliminating u_2 from (4.45), (4.46) we have the second-order differential equation

$$u_{1xx} - 2\alpha u_{1x} + (\alpha^2 + \beta^2)u_1 = 0.$$

This equation has general solutions as

$$u_1 = e^{\alpha x}[D_1(t)\cos\beta x + D_2(t)\sin\beta x], \tag{4.53}$$

where D_1, D_2 do not depend on x.
After substituting u_1 into (4.45), we get

$$u_2 = e^{\alpha x}[D_1(t)\sin\beta x - D_2(t)\cos\beta x]. \tag{4.54}$$

From (4.45), (4.46) we have

$$u_{1xxx} = \alpha(\alpha^2 - 3\beta^2)u_1 - \beta(3\alpha^2 - \beta^2)u_2,$$
$$u_{2xxx} = \beta(3\alpha^2 - \beta^2)u_1 + \alpha(\alpha^2 - 3\beta^2)u_2.$$

From these equations and (4.47), (4.48) we have the system

$$D_{1t} = -4\alpha(\alpha^2 - 3\beta^2)D_1 - 4\beta(3\alpha^2 - \beta^2)D_2, \tag{4.55}$$

$$D_{2t} = 4\beta(3\alpha^2 - \beta^2)D_1 - 4\alpha(\alpha^2 - 3\beta^2)D_2. \tag{4.56}$$

The general solutions of (4.55), (4.56) are of the form

$$D_1(t) = e^{-4\alpha(\alpha^2-3\beta^2)t}\{C_1\cos[4\beta(3\alpha^2 - \beta^2)t] + C_2\sin[4\beta(3\alpha^2 - \beta^2)t]\}, \tag{4.57}$$

$$D_2(t) = e^{-4\alpha(\alpha^2-3\beta^2)t}\{C_1\sin[4\beta(3\alpha^2 - \beta^2)t] - C_2\cos[4\beta(3\alpha^2 - \beta^2)t]\}, \tag{4.58}$$

where C_1, C_2 are two arbitrary real constants.
Thereby, we have the general solution of system (4.45)-(4.48)

(4.59)
$$u_1 = e^{\alpha x-4\alpha(\alpha^2-3\beta^2)t}\{C_1\cos[\beta x - 4\beta(3\alpha^2 - \beta^2)t] - C_2\sin[\beta x - 4\beta(3\alpha^2 - \beta^2)t]\},$$

(4.60)
$$u_2 = e^{\alpha x-4\alpha(\alpha^2-3\beta^2)t}\{C_1\sin[\beta x - 4\beta(3\alpha^2 - \beta^2)t] + C_2\cos[\beta x - 4\beta(3\alpha^2 - \beta^2)t]\}.$$

Similarly, from (4.49)-(4.52) we have

(4.61)
$$v_1 = e^{-\alpha x+4\alpha(\alpha^2-3\beta^2)t}\{C_3\cos[\beta x - 4\beta(3\alpha^2 - \beta^2)t] + C_4\sin[\beta x - 4\beta(3\alpha^2 - \beta^2)t]\},$$

(4.62)
$$v_2 = e^{-\alpha x+4\alpha(\alpha^2-3\beta^2)t}\{-C_3\sin[\beta x - 4\beta(3\alpha^2 - \beta^2)t] + C_4\cos[\beta x - 4\beta(3\alpha^2 - \beta^2)t]\},$$

where C_3, C_4 are two arbitrary real constants.

From (4.59)-(4.62), it follows that $(4.43),(4.44)$ are a solution of the system $(4.39)-(4.42)$. □

Next, from $(4.43),(4.44)$ we have

$$\begin{aligned} f &= W(\phi_1,\phi_2) \\ &= \beta\big[(C_1^2+C_2^2)e^{2\alpha x-8\alpha(\alpha^2-3\beta^2)t}+(C_3^2+C_4^2)e^{-2\alpha x+8\alpha(\alpha^2-3\beta^2)t}\big] \\ &\quad + 2i\alpha\big\{(C_2C_3-C_1C_4)\cos[2\beta x-8\beta(\beta^2-3\alpha^2)t] \\ &\quad + (C_1C_3+C_2C_4)\sin[2\beta x-8\beta(\beta^2-3\alpha^2)t]\big\}. \end{aligned} \tag{4.63}$$

Thus, the solution u of the mKdV equation is given as follows:

$$\begin{aligned} u =& 2\frac{\partial}{\partial x}\arctan\Big\{\Big[\beta\big\{(C_1^2+C_2^2)e^{2\alpha x-8\alpha(\alpha^2-3\beta^2)t}+(C_3^2+C_4^2)e^{-2\alpha x+8\alpha(\alpha^2-3\beta^2)t}\big\}\Big]^{-1} \\ &\times\Big[2\alpha\big\{(C_2C_3-C_1C_4)\cos[2\beta x-8\beta(\beta^2-3\alpha^2)t] \\ &+(C_1C_3+C_2C_4)\sin[2\beta x-8\beta(\beta^2-3\alpha^2)t]\big\}\Big]\Big\}. \end{aligned} \tag{4.64}$$

We note that in this case the Wronskian solution (4.63) depends on 4 arbitrary real numbers.

4.4. **Case: Γ is constructed from several 2×2 real Jordan blocks.** We consider the case that $N=m=2n$, in which the matrix $\Gamma=\Gamma_{2n}$ is of the form

$$\Gamma_{2n}=\begin{pmatrix}\Gamma_{1,2} & O & \dots & O\\ O & \Gamma_{2,2} & \dots & O\\ \vdots & \vdots & \ddots & \vdots\\ O & O & \dots & \Gamma_{n,2}\end{pmatrix}, \tag{4.65}$$

where O is 2×2 zero matrix and $\Gamma_{j,2}=\begin{pmatrix}\alpha_j & -\beta_j\\ \beta_j & \alpha_j\end{pmatrix}$, $\beta_j\neq 0$.

The matrix Γ_{2n} in (4.65) corresponds to the composition of n pairs of complex eigenvalues $\lambda_j=\alpha_j+i\beta_j, \bar{\lambda}_j=\alpha_j-i\beta_j, j=1,2,\dots,n$. In this case, the condition equations can be separated into n independent systems that had been considered in Subsection 4.3. Then, by virtue of Theorem 4.2, the Wronskian solution is given as

$$f_{2n}=W(\phi_{11},\phi_{12},\dots,\phi_{n1},\phi_{n2}), \tag{4.66}$$

where for $j = 1, 2, \dots, n$ it holds

(4.67)
$$\phi_{j1} = e^{\alpha_j x - 4\alpha_j(\alpha_j^2 - 3\beta_j^2)t}\{C_{j1}\cos[\beta_j x - 4\beta_j(3\alpha_j^2 - \beta_j^2)t] - C_{j2}\sin[\beta_j x - 4\beta_j(3\alpha_j^2 - \beta_j^2)t]\}$$
$$+ ie^{-\alpha_j x + 4\alpha_j(\alpha_j^2 - 3\beta_j^2)t}\{C_{j3}\cos[\beta_j x - 4\beta_j(3\alpha_j^2 - \beta_j^2)t] + C_{j4}\sin[\beta_j x - 4\beta_j(3\alpha_j^2 - \beta_j^2)t]\},$$
(4.68)
$$\phi_{j2} = e^{\alpha_j x - 4\alpha_j(\alpha_j^2 - 3\beta_j^2)t}\{C_{j1}\sin[\beta_j x - 4\beta_j(3\alpha_j^2 - \beta_j^2)t] + C_{j2}\cos[\beta_j x - 4\beta_j(3\alpha_j^2 - \beta_j^2)t]\}$$
$$+ ie^{-\alpha_j x + 4\alpha_j(\alpha_j^2 - 3\beta_j^2)t}\{-C_{j3}\sin[\beta_j x - 4\beta_j(3\alpha_j^2 - \beta_j^2)t] + C_{j4}\cos[\beta_j x - 4\beta_j(3\alpha_j^2 - \beta_j^2)t]\},$$

and $C_{j1}, C_{j2}, C_{j3}, C_{j4}$ are arbitrary real constants.
Now, we put $g_j = e^{\alpha_j x - 4\alpha_j(\alpha_j^2 - 3\beta_j^2)t}$, $\eta_j = \beta_j x - 4\beta_j(3\alpha_j^2 - \beta_j^2)t$, and then (4.67)(4.68) can be rewritten in the form

$$\phi_{j1} = g_j[C_{j1}\cos\eta_j - C_{j2}\sin\eta_j] + ig_j^{-1}[C_{j3}\cos\eta_j + C_{j4}\sin\eta_j],$$
$$\phi_{j2} = g_j[C_{j1}\sin\eta_j + C_{j2}\cos\eta_j] + ig_j^{-1}[-C_{j3}\sin\eta_j + C_{j4}\cos\eta_j].$$

By direct computation we can show that for $k = 0, 1, 2, \dots$ it holds

(4.69)
$$\phi_{j1}^{(k)} = g_j[D_{j1}(k)\cos\eta_j - D_{j2}(k)\sin\eta_j] + ig_j^{-1}[D_{j3}(k)\cos\eta_j + D_{j4}(k)\sin\eta_j],$$
(4.70)
$$\phi_{j2}^{(k)} = g_j[D_{j1}(k)\sin\eta_j + D_{j2}(k)\cos\eta_j] + ig_j^{-1}[-D_{j3}(k)\sin\eta_j + D_{j4}(k)\cos\eta_j],$$

where the coefficients $D_{jl}(k)$ are given by the following recursion relations:

$$D_{j1}(k) = \alpha_j D_{j1}(k-1) - \beta_j D_{j2}(k-1), \tag{4.71}$$
$$D_{j2}(k) = \beta_j D_{j1}(k-1) + \alpha_j D_{j2}(k-1), \tag{4.72}$$
$$D_{j3}(k) = -\alpha_j D_{j3}(k-1) + \beta_j D_{j4}(k-1), \tag{4.73}$$
$$D_{j4}(k) = -\beta_j D_{j3}(k-1) - \alpha_j D_{j4}(k-1) \tag{4.74}$$

and $D_{jl}(0) = C_{jl}$ for $j = 1, 2, \dots, n;\ l = 1, 2, 3, 4$.

Note that in case $C_{j1}^2 + C_{j2}^2 \neq 0,\ C_{j3}^2 + C_{j4}^2 \neq 0$ for $j = 1, 2, \dots, n$ the difference equations (4.71)-(4.74) can be solved as follows:

$$D_{j1}(k) = \sqrt{C_{j1}^2 + C_{j2}^2}\sqrt{(\alpha_j^2 + \beta_j^2)^k}\cos(\xi_{j1} + k\zeta_j),$$
$$D_{j2}(k) = \sqrt{C_{j1}^2 + C_{j2}^2}\sqrt{(\alpha_j^2 + \beta_j^2)^k}\sin(\xi_{j1} + k\zeta_j),$$
$$D_{j3}(k) = (-1)^k\sqrt{C_{j3}^2 + C_{j4}^2}\sqrt{(\alpha_j^2 + \beta_j^2)^k}\cos(\xi_{j2} + k\zeta_j),$$
$$D_{j4}(k) = (-1)^k\sqrt{C_{j3}^2 + C_{j4}^2}\sqrt{(\alpha_j^2 + \beta_j^2)^k}\sin(\xi_{j2} + k\zeta_j),$$

where

$$\cos\xi_{j1} = \frac{C_{j1}}{\sqrt{C_{j1}^2 + C_{j2}^2}}, \quad \cos\xi_{j2} = \frac{C_{j3}}{\sqrt{C_{j3}^2 + C_{j4}^2}}, \quad \cos\zeta_j = \frac{\alpha_j}{\sqrt{\alpha_j^2 + \beta_j^2}},$$
$$\sin\xi_{j1} = \frac{C_{j2}}{\sqrt{C_{j1}^2 + C_{j2}^2}}, \quad \sin\xi_{j2} = \frac{C_{j4}}{\sqrt{C_{j3}^2 + C_{j4}^2}}, \quad \sin\zeta_j = \frac{\beta_j}{\sqrt{\alpha_j^2 + \beta_j^2}}.$$

For any k if $C_{j1} = C_{j2} = 0$ then $D_{j1}(k) = D_{j2}(k) = 0$, and if $C_{j3} = C_{j4} = 0$ then $D_{j3}(k) = D_{j4}(k) = 0$.

Next, (4.69), (4.70) can be rewritten in the form

(4.75)
$$\phi_{j1} = \sum_{\epsilon_{j1}=\pm 1} \big\{[D_{j1}(k)\cos\eta_j - D_{j2}(k)\sin\eta_j]\frac{1+\epsilon_{j1}}{2} + i[D_{j3}(k)\cos\eta_j + D_{j4}(k)\sin\eta_j]\frac{1-\epsilon_{j1}}{2}\big\}g^{\epsilon_{j1}},$$

(4.76)
$$\phi_{j2} = \sum_{\epsilon_{j2}=\pm 1} \big\{[D_{j1}(k)\sin\eta_j + D_{j2}(k)\cos\eta_j]\frac{1+\epsilon_{j2}}{2} + i[-D_{j3}(k)\sin\eta_j + D_{j4}(k)\cos\eta_j]\frac{1-\epsilon_{j2}}{2}\big\}g^{\epsilon_{j2}}.$$

We denote by $D(\epsilon_{11}, \epsilon_{12}, \dots, \epsilon_{n1}, \epsilon_{n2})$ the $2n \times 2n$ determinant whose entries of the $(2j-1)$-th row and the $(2j)$-th row for each j are respectively equal to

$$\Big([D_{j1}(k)\cos\eta_j - D_{j2}(k)\sin\eta_j]\frac{1+\epsilon_{j1}}{2} + i[D_{j3}(k)\cos\eta_j + D_{j4}(k)\sin\eta_j]\frac{1-\epsilon_{j1}}{2}\Big), \quad k = 0, 1, 2, \dots, 2n-1,$$
$$\Big([D_{j1}(k)\sin\eta_j + D_{j2}(k)\cos\eta_j]\frac{1+\epsilon_{j2}}{2} + i[-D_{j3}(k)\sin\eta_j + D_{j4}(k)\cos\eta_j]\frac{1-\epsilon_{j2}}{2}\Big), \quad k = 0, 1, 2, \dots, 2n-1.$$

Then we can write the Wronskian (4.66) in the form

$$f_{2n} = \sum_{(\epsilon_{11}, \epsilon_{12}, \dots, \epsilon_{n1}, \epsilon_{n2})} \Big[\Big(\prod_{j=1}^{n} g_j^{\epsilon_{j1}+\epsilon_{j2}}\Big) D(\epsilon_{11}, \epsilon_{12}, \dots, \epsilon_{n1}, \epsilon_{n2})\Big]. \tag{4.77}$$

Note that for $l = 1$ or 2 all entries of the $(2j+l-2)$-th row of the determinant $D(\epsilon_{11}, \epsilon_{12}, \dots, \epsilon_{n1}, \epsilon_{n2})$ are real-valued if $\epsilon_{jl} = 1$, and they are pure-imaginary-valued if $\epsilon_{jl} = -1$. Consequently, if $\prod_{j=1}^{n}(\epsilon_{j1}.\epsilon_{j2}) = 1$ then the determinant $D(\epsilon_{11}, \epsilon_{12}, \dots, \epsilon_{n1}, \epsilon_{n2})$ is real-valued, and if otherwise, then it is pure-imaginary-valued. We define the sets X_1, X_2 as in Subsection 4.1. From (4.77) we can obtain a class of Wronskian solutions in the following form:

$$f_{2n} = F_{2n} + iG_{2n}, \tag{4.78}$$

where F_{2n}, G_{2n} are respectively real and imaginary parts of the Wronskian (4.77) and are given by

$$F_{2n} = \sum_{(\epsilon_{11},\epsilon_{12},\dots,\epsilon_{n1},\epsilon_{n2})\in X_1} \Big[\Big(\prod_{j=1}^{n} g_j^{\epsilon_{j1}+\epsilon_{j2}}\Big) D(\epsilon_{11},\epsilon_{12},\dots,\epsilon_{n1},\epsilon_{n2})\Big], \tag{4.79}$$

$$G_{2n} = (-i) \sum_{(\epsilon_{11},\epsilon_{12},\dots,\epsilon_{n1},\epsilon_{n2})\in X_2} \Big[\Big(\prod_{j=1}^{n} g_j^{\epsilon_{j1}+\epsilon_{j2}}\Big) D(\epsilon_{11},\epsilon_{12},\dots,\epsilon_{n1},\epsilon_{n2})\Big]. \tag{4.80}$$

In this case, the Wronskian solutions depend on $4n$ arbitrary real constants. In particular, if $n = 1$ then we have the solution (4.63).

In the case that $n = 2$, it is complicated to compute f_4 by using (4.78)-(4.80) since one has to compute 16 determinants $D(\epsilon_{11}, \epsilon_{12}, \epsilon_{21}, \epsilon_{22})$ of order 4. To avoid this process, we propose another way to compute f_4. From (4.66) we have

$$f_4 = \begin{vmatrix} \phi_{11} & \phi_{11x} & \phi_{11xx} & \phi_{11xxx} \\ \phi_{12} & \phi_{12x} & \phi_{12xx} & \phi_{12xxx} \\ \phi_{21} & \phi_{21x} & \phi_{21xx} & \phi_{21xxx} \\ \phi_{22} & \phi_{22x} & \phi_{22xx} & \phi_{22xxx} \end{vmatrix}. \tag{4.81}$$

By using the Laplace expansion, the determinant (4.81) can be expressed in terms of 2 by 2 minors. After some calculations, we can rewrite f_4 in the form

(4.82)
$$\begin{aligned} f_4 =&\Big\{\beta_1\beta_2\big[(\alpha_1^2+\alpha_2^2+\beta_1^2+\beta_2^2)^2 + 4(\alpha_1^2\alpha_2^2 - \beta_1^2\beta_2^2)\big](e_{11}+e_{12})(e_{21}+e_{22}) \\ &- 4\alpha_1\beta_1\alpha_2\beta_2(\alpha_1^2+\alpha_2^2+\beta_1^2+\beta_2^2)[(e_{11}-e_{12})(e_{21}-e_{22}) + 4e_{13}e_{23}] \\ &- 4\alpha_1\alpha_2\big[(\alpha_1^2+\alpha_2^2+\beta_1^2+\beta_2^2)^2 - 4(\alpha_1^2\alpha_2^2 - \beta_1^2\beta_2^2)\big]e_{14}e_{24}\Big\} \\ &+ i\Big\{2\alpha_2\beta_1[(\alpha_1^2+\beta_1^2-\alpha_2^2-\beta_2^2)^2 + 4(\beta_1^2\alpha_2^2 - \alpha_1^2\beta_2^2)](e_{11}+e_{12})e_{24} \\ &+ 2\alpha_1\beta_2[(\alpha_1^2+\beta_1^2-\alpha_2^2-\beta_2^2)^2 - 4(\beta_1^2\alpha_2^2 - \alpha_1^2\beta_2^2)](e_{21}+e_{22})e_{14} \\ &+ 8\alpha_1\beta_1\alpha_2\beta_2(\alpha_1^2+\beta_1^2-\alpha_2^2-\beta_2^2)[(e_{21}-e_{22})e_{13} - (e_{11}-e_{12})e_{23}]\Big\}, \end{aligned}$$

where the auxiliary functions e_{jl} are given by

$$\begin{aligned} e_{j1} =&(C_{j1}^2 + C_{j2}^2)g_j^2, \quad e_{j2} = (C_{j3}^2 + C_{j3}^2)g_j^{-2}, \\ e_{j3} =&(C_{j1}C_{j3} + C_{j2}C_{j4})\cos(2\eta_j) - (C_{j2}C_{j3} - C_{j1}C_{j4})\sin(2\eta_j), \\ e_{j4} =&(C_{j2}C_{j3} - C_{j1}C_{j4})\cos(2\eta_j) + (C_{j1}C_{j3} + C_{j2}C_{j4})\sin(2\eta_j), \end{aligned}$$

for $j = 1, 2$.

4.5. **Case: Γ_m is a real Jordan block with a pair of multiple complex eigenvalues.** We consider the case that Γ_m in (4.1) correspond to two complex

Jordan blocks

$$(4.83)\qquad \begin{pmatrix} \lambda & & & 0 \\ 1 & \lambda & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda \end{pmatrix}_{n\times n} \quad \text{and} \quad \begin{pmatrix} \bar\lambda & & & 0 \\ 1 & \bar\lambda & & \\ & \ddots & \ddots & \\ 0 & & 1 & \bar\lambda \end{pmatrix}_{n\times n},$$

where $\lambda = \alpha + i\beta$, $\bar\lambda = \alpha - i\beta$, $\beta \neq 0$ and $m = 2n$.
In this case, the matrix Γ_{2n} is of the form

$$(4.84)\qquad \Gamma_{2n} = \begin{pmatrix} U & O & O & \dots & O & O \\ I & U & O & \dots & O & O \\ O & I & U & \dots & O & O \\ \dots & \dots & \dots & \dots & \dots & \dots \\ O & O & O & \dots & I & U \end{pmatrix}_{2n\times 2n},$$

where $U = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and O is 2×2 zero matrix.
We need to use the following $2n \times 2n$ lower triangular matrix consisting 2×2 blocks of operators:

$$(4.85)\qquad V = \begin{pmatrix} I & O & O & \dots & O & O \\ V_1 & I & O & \dots & O & O \\ V_2 & V_1 & I & \dots & O & O \\ \dots & \dots & \dots & \dots & \dots & \dots \\ V_{n-1} & V_{n-2} & V_{n-3} & \dots & V_1 & I \end{pmatrix}_{2n\times 2n},$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad V_j = \begin{pmatrix} \frac{1}{j!}\frac{\partial^j}{\partial\alpha^j} & 0 \\ 0 & \frac{1}{j!}\frac{\partial^j}{\partial\alpha^j} \end{pmatrix}.$$

For convenience we put $\phi = (\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22}, \dots, \phi_{n1}, \phi_{n2})^T$.

Theorem 4.3. *The general solutions Φ of system* (4.1), (4.2) *with the Jordan block Γ_{2n} in* (4.84) *are of the form*

$$(4.86)\qquad \phi = V\tilde\phi,$$

where $\tilde\phi = (\tilde\phi_{11}, \tilde\phi_{12}, \tilde\phi_{21}, \tilde\phi_{22}, \dots, \tilde\phi_{n1}, \tilde\phi_{n2})^T$,

(4.87)

$$\begin{aligned}\tilde\phi_{j1} =& e^{\alpha x - 4\alpha(\alpha^2-3\beta^2)t}\{C_{j1}\cos[\beta x - 4\beta(3\alpha^2-\beta^2)t] - C_{j2}\sin[\beta x - 4\beta(3\alpha^2-\beta^2)t]\} \\ &+ ie^{-\alpha x + 4\alpha(\alpha^2-3\beta^2)t}\{C_{j3}\cos[\beta x - 4\beta(3\alpha^2-\beta^2)t] + C_{j4}\sin[\beta x - 4\beta(3\alpha^2-\beta^2)t]\},\end{aligned}$$

(4.88)

$$\begin{aligned}\tilde\phi_{j2} =& e^{\alpha x - 4\alpha(\alpha^2-3\beta^2)t}\{C_{j1}\sin[\beta x - 4\beta(3\alpha^2-\beta^2)t] + C_{j2}\cos[\beta x - 4\beta(3\alpha^2-\beta^2)t]\} \\ &+ ie^{-\alpha x + 4\alpha(\alpha^2-3\beta^2)t}\{-C_{j3}\sin[\beta x - 4\beta(3\alpha^2-\beta^2)t] + C_{j4}\cos[\beta x - 4\beta(3\alpha^2-\beta^2)t]\},\end{aligned}$$

and $C_{j1}, C_{j2}, C_{j3}, C_{j4}$, $j = 1, 2, \dots, n$ are arbitrary real constants.

Proof. In this case, using the transformation

$$\begin{aligned}
\phi_{11} &= \tilde{\phi}_{11},\\
\phi_{12} &= \tilde{\phi}_{12},\\
\phi_{21} &= \tilde{\phi}_{21} + \frac{\partial}{\partial\alpha}\tilde{\phi}_{11},\\
\phi_{22} &= \tilde{\phi}_{22} + \frac{\partial}{\partial\alpha}\tilde{\phi}_{12},\\
\vdots \quad & \quad \vdots\\
\phi_{n1} &= \tilde{\phi}_{n1} + \frac{\partial}{\partial\alpha}\tilde{\phi}_{n-1,1} + \cdots + \frac{1}{(n-1)!}\frac{\partial^{n-1}}{\partial\alpha^{n-1}}\tilde{\phi}_{11},\\
\phi_{n2} &= \tilde{\phi}_{n2} + \frac{\partial}{\partial\alpha}\tilde{\phi}_{n-1,2} + \cdots + \frac{1}{(n-1)!}\frac{\partial^{n-1}}{\partial\alpha^{n-1}}\tilde{\phi}_{12}
\end{aligned}$$

we obtain

$$\tilde{\phi}_{j1x} = \alpha\overline{\tilde{\phi}_{j1}} - \beta\overline{\tilde{\phi}_{j2}}, \tag{4.89}$$

$$\tilde{\phi}_{j2x} = \beta\overline{\tilde{\phi}_{j1}} + \alpha\overline{\tilde{\phi}_{j2}}, \tag{4.90}$$

$$\tilde{\phi}_{j1t} = -4\tilde{\phi}_{j1xxx}, \tag{4.91}$$

$$\tilde{\phi}_{j2t} = -4\tilde{\phi}_{j2xxx}, \tag{4.92}$$

for $j = 1, 2, \ldots, n$.
By virtue of Theorem 4.2 and from (4.89)-(4.92) it follows that (4.87), (4.88) are general solutions of (4.1), (4.2). □

Now we give an explicit representation for the Wronskian

$$f_{2n} = W(\phi_{11}, \phi_{12}, \ldots, \phi_{n1}, \phi_{n2}), \tag{4.93}$$

where ϕ is determined by (4.86).
We put $g = e^{\alpha x - 4\alpha(\alpha^2 - 3\beta^2)t}$ and $\eta = \beta x - 4\beta(3\alpha^2 - \beta^2)t$. By (4.86)-(4.88), the functions ϕ_{j1}, ϕ_{j2} can be rewritten in the forms

$$\phi_{j1} = g\big[P_{j1}\cos\eta - P_{j2}\sin\eta\big] + ig^{-1}\big[P_{j3}\cos\eta + P_{j4}\sin\eta\big], \tag{4.94}$$

$$\phi_{j2} = g\big[P_{j1}\sin\eta + P_{j2}\cos\eta\big] + ig^{-1}\big[-P_{j3}\sin\eta + P_{j4}\cos\eta\big], \tag{4.95}$$

where $P_{jl} \equiv P_{jl}(x, t, \alpha, \beta)$, $j = 1, 2, \ldots, n$; $l = 1, 2, 3, 4$ are some polynomials of variables x, t, α, β and can be determined from the following linear algebraic systems

$$\begin{cases} g\big[P_{j1}\cos\eta - P_{j2}\sin\eta\big] = \sum_{l=1}^{j}\Big[C_{l1}\frac{1}{(j-l)!}\frac{\partial^{j-l}}{\partial\alpha^{j-l}}(g\cos\eta) - C_{l2}\frac{1}{(j-l-1)!}\frac{\partial^{j-l}}{\partial\alpha^{j-l}}(g\sin\eta)\Big], \\ g\big[P_{j1}\sin\eta + P_{j2}\cos\eta\big] = \sum_{l=1}^{j}\Big[C_{l1}\frac{1}{(j-l)!}\frac{\partial^{j-l}}{\partial\alpha^{j-l}}(g\sin\eta) + C_{l2}\frac{1}{(j-l)!}\frac{\partial^{j-l}}{\partial\alpha^{j-l}}(g\cos\eta)\Big], \end{cases}$$

$$\begin{cases} g^{-1}\big[P_{j3}\cos\eta + P_{j4}\sin\eta\big] = \sum_{l=1}^{j}\Big[C_{l3}\frac{1}{(j-l)!}\frac{\partial^{j-l}}{\partial\alpha^{j-l}}(g^{-1}\cos\eta) + C_{l4}\frac{1}{(j-l)!}\frac{\partial^{j-l}}{\partial\alpha^{j-l}}(g^{-1}\sin\eta)\Big], \\ g^{-1}\big[-P_{j3}\sin\eta + P_{j4}\cos\eta\big] = \sum_{l=1}^{j}\Big[-C_{l3}\frac{1}{(j-l)!}\frac{\partial^{j-l}}{\partial\alpha^{j-l}}(g^{-1}\sin\eta) + C_{l4}\frac{1}{(j-l)!}\frac{\partial^{j-l}}{\partial\alpha^{j-l}}(g^{-1}\cos\eta)\Big]. \end{cases}$$

Next, we see that the derivatives $\phi_{j1}^{(k)}$ and $\phi_{j2}^{(k)}$ of ϕ_{j1}, ϕ_{j2} can be written in the form

(4.96)
$$\phi_{j1}^{(k)} = g\big[Q_{j1}(k)\cos\eta - Q_{j2}(k)\sin\eta\big] + ig^{-1}\big[Q_{j3}(k)\cos\eta + Q_{j4}(k)\sin\eta\big],$$
(4.97)
$$\phi_{j2}^{(k)} = g\big[Q_{j1}(k)\sin\eta + Q_{j2}(k)\cos\eta\big] + ig^{-1}\big[-Q_{j3}(k)\sin\eta + Q_{j4}(k)\cos\eta\big],$$

where the polynomials $Q_{jl}(k) \equiv Q_{jl}(x,t,k)$ are given by the following recursion relations

(4.98) $$Q_{j1}(k) = \alpha Q_{j1}(k-1) - \beta Q_{j2}(k-1) + Q_{j1x}(k-1),$$
(4.99) $$Q_{j2}(k) = \beta Q_{j1}(k-1) + \alpha Q_{j2}(k-1) + Q_{j2x}(k-1),$$
(4.100) $$Q_{j3}(k) = -\alpha Q_{j3}(k-1) + \beta Q_{j4}(k-1) + Q_{j3x}(k-1),$$
(4.101) $$Q_{j4}(k) = -\beta Q_{j3}(k-1) - \alpha Q_{j4}(k-1) + Q_{j4x}(k-1),$$
$$Q_{jl}(0) = Q_{jl}(x,t,0) = P_{jl}(x,t),$$

for $j = 1, 2, \dots, n; l = 1, 2, 3, 4.$
Then we can rewrite (4.96), (4.97) as follows:

(4.102)
$$\phi_{j1}^{(k)} = \sum_{\epsilon_{j1}=\pm1}\Big\{\big[Q_{j1}(k)\cos\eta - Q_{j2}(k)\sin\eta\big]\frac{1+\epsilon_{j1}}{2} + i\big[Q_{j3}(k)\cos\eta + Q_{j4}(k)\sin\eta\big]\frac{1-\epsilon_{j1}}{2}\Big\}g^{\epsilon_{j1}},$$

(4.103)
$$\phi_{j2}^{(k)} = \sum_{\epsilon_{j2}=\pm1}\Big\{\big[Q_{j1}(k)\sin\eta + Q_{j2}(k)\cos\eta\big]\frac{1+\epsilon_{j2}}{2} + i\big[-Q_{j3}(k)\sin\eta + Q_{j4}(k)\cos\eta\big]\frac{1-\epsilon_{j2}}{2}\Big\}g^{\epsilon_{j2}}.$$

We denote by $D(\epsilon_{11}, \epsilon_{12}, \dots, \epsilon_{n1}, \epsilon_{n2})$ the $2n \times 2n$ determinant whose entries of the $(2j-1)$-th and the $(2j)$-th rows for each j are respectively equal to

$$\Big(\big\{\big[Q_{j1}(k)\cos\eta - Q_{j2}(k)\sin\eta\big]\frac{1+\epsilon_{j1}}{2} + i\big[Q_{j3}(k)\cos\eta + Q_{j4}(k)\sin\eta\big]\frac{1-\epsilon_{j1}}{2}\big\}\Big), \quad k = 0, 1, 2, \dots, 2n-1,$$

$$\Big(\big\{\big[Q_{j1}(k)\sin\eta + Q_{j2}(k)\cos\eta\big]\frac{1+\epsilon_{j2}}{2} + i\big[-Q_{j3}(k)\sin\eta + Q_{j4}(k)\cos\eta\big]\frac{1-\epsilon_{j2}}{2}\big\}\Big), \quad k = 0, 1, 2, \dots, 2n-1.$$

Now, the Wronskian (4.93) can be transformed into the form

$$f_{2n} = \sum_{(\epsilon_{11}, \epsilon_{12}, \dots, \epsilon_{n1}, \epsilon_{n2})} \Big[\Big(\prod_{j=1}^{n} g^{\epsilon_{j1}+\epsilon_{j2}}\Big) D(\epsilon_{11}, \epsilon_{12}, \dots, \epsilon_{n1}, \epsilon_{n2})\Big]. \tag{4.104}$$

Note that for $l = 1$ or 2 all entries of the $(2n+l-2)$-th row of the determinant $D(\epsilon_{11}, \epsilon_{12}, \dots, \epsilon_{n1}, \epsilon_{n2})$ are real-valued if $\epsilon_{jl} = 1$, and they are pure-imaginary-valued if $\epsilon_{jl} = -1$. Consequently, the value of the determinant $D(\epsilon_{11}, \epsilon_{12}, \dots, \epsilon_{n1}, \epsilon_{n2})$ is real-valued if $\prod_{j=1}^{n}(\epsilon_{j1}.\epsilon_{j2}) = 1$, and else it is pure-imaginary-valued. We say that the m-tuple $(\epsilon_{11}, \epsilon_{12}, \dots, \epsilon_{n1}, \epsilon_{n2})$ belongs to X_1 if $\prod_{j=1}^{n}(\epsilon_{j1}.\epsilon_{j2}) = 1$. Otherwise, it belongs to X_2. Thus, from (4.104) we obtain a class of the Wronskian solutions in the form

$$f_{2n} = F_{2n} + iG_{2n}, \tag{4.105}$$

where F_{2n}, G_{2n} are respectively real and imaginary parts of the Wronskian (4.104) and are given by the relations

$$F_{2n} = \sum_{(\epsilon_{11}, \epsilon_{12}, \dots, \epsilon_{n1}, \epsilon_{n2}) \in X_1} \Big[\Big(\prod_{j=1}^{n} g^{\epsilon_{j1}+\epsilon_{j2}}\Big) D(\epsilon_{11}, \epsilon_{12}, \dots, \epsilon_{n1}, \epsilon_{n2})\Big], \tag{4.106}$$

$$G_{2n} = (-i) \sum_{(\epsilon_{11}, \epsilon_{12}, \dots, \epsilon_{n1}, \epsilon_{n2}) \in X_2} \Big[\Big(\prod_{j=1}^{n} g^{\epsilon_{j1}+\epsilon_{j2}}\Big) D(\epsilon_{11}, \epsilon_{12}, \dots, \epsilon_{n1}, \epsilon_{n2})\Big]. \tag{4.107}$$

The solutions of (4.105) depend on $4n$ arbitrary real constants. Note that if $n = 1$ we have the solution (4.63).

In particular, if we choose $C_{j1} = C_{j2} = C_{j3} = C_{j4} = 0$ for $j \geq 2$, then we have the bi-directional Wronskian solutions

$$f_{2n} = W\Big(\phi_{11}, \phi_{12}, \frac{\partial}{\partial\alpha}\phi_{11}, \frac{\partial}{\partial\alpha}\phi_{12}, \dots, \frac{1}{(n-1)!}\frac{\partial^{n-1}}{\partial\alpha^{n-1}}\phi_{11}, \frac{1}{(n-1)!}\frac{\partial^{n-1}}{\partial\alpha^{n-1}}\phi_{12}\Big). \tag{4.108}$$

When $N = 2n = 4$ the Wronskian determinant (4.108) becomes

$$f_4 = W(\phi_{11}, \phi_{12}, \phi_{11\alpha}, \phi_{12\alpha}) = \begin{vmatrix} \phi_{11} & \phi_{11x} & \phi_{11xx} & \phi_{11xxx} \\ \phi_{12} & \phi_{12x} & \phi_{12xx} & \phi_{12xxx} \\ \phi_{11\alpha} & \phi_{11\alpha x} & \phi_{11\alpha xx} & \phi_{11\alpha xxx} \\ \phi_{12\alpha} & \phi_{12\alpha x} & \phi_{12\alpha xx} & \phi_{12\alpha xxx} \end{vmatrix} \tag{4.109}$$

where ϕ_{11}, ϕ_{12} are given in (4.87), (4.88).
In order to avoid the complicated computation that had been described at the end of subsection 4.4, we propose here another way to compute f_4 instead of using (4.105)-(4.108). By direct computing, the α-direction derivatives $\phi_{11\alpha}$ and $\phi_{12\alpha}$ can be written in the form

$$\phi_{11\alpha} = \big[x - 12(\alpha^2 - \beta^2)t\big]\overline{\phi}_{11} + 24\alpha\beta t\overline{\phi}_{12}, \tag{4.110}$$

$$\phi_{12\alpha} = -24\alpha\beta t\overline{\phi}_{11} + \big[x - 12(\alpha^2 - \beta^2)t\big]\overline{\phi}_{12}. \tag{4.111}$$

By virtue of the Laplace expansion the determinant (4.109) can be expressed in terms of 2×2 minors. By using equations (4.1), (4.2), (4.110), (4.111), after some calculations, we can rewrite the Wronskian f_4 in the explicit form

(4.112)
$$\begin{aligned} f_4 = \Big\{ &\big[8\alpha^2\beta^2(\alpha^2+\beta^2)R - 12\alpha^2\beta^2 + 4\beta^4\big](e_1+e_2)^2 \\ &- \big[8\alpha^2\beta^2(\alpha^2+\beta^2)R + 12\alpha^2\beta^2\big](e_1-e_2)^2 + \big[32\alpha^2\beta^2(\alpha^2+\beta^2)R \\ &- 48\alpha^2\beta^2\big]e_3^2 + \big[32\alpha^2\beta^2(\alpha^2+\beta^2)R + 48\alpha^2\beta^2 - 16\alpha^4\big]e_4^2\Big\} \\ &+ i\Big\{32\alpha^2\beta^2(-\alpha h_1 + \beta h_2)(e_1+e_2)e_3 - 32\alpha^2\beta^2(\beta h_1 + \alpha h_2)(e_1-e_2)e_4 \\ &- 32\alpha\beta(\alpha^2+\beta^2)(e_1+e_2)e_4\Big\}, \end{aligned}$$

where

$$\begin{aligned} e_1 &= (C_{11}^2 + C_{12}^2)g^2, \quad e_2 = (C_{13}^2 + C_{14}^2)g^{-2}, \\ e_3 &= (C_{11}C_{13} + C_{12}C_{14})\cos(2\eta) - (C_{12}C_{13} - C_{11}C_{14})\sin(2\eta), \\ e_4 &= (C_{12}C_{13} - C_{11}C_{14})\cos(2\eta) + (C_{11}C_{13} + C_{12}C_{14})\sin(2\eta), \\ h_1 &= x - 12(\alpha^2-\beta^2)t, \quad h_2 = -24\alpha\beta t, \quad R = h_1^2 + h_2^2. \end{aligned}$$

Note that the Wronskian f_4 contains only 4 arbitrary real numbers.

Remark 2. We can also consider the case that the matrix Γ_m in system (4.1), (4.2) is of the form

$$\Gamma_m = \begin{pmatrix} U & O & O & \dots & O & O \\ \Sigma & U & O & \dots & O & O \\ O & \Sigma & U & \dots & O & O \\ \dots & \dots & \dots & \dots & \dots & \dots \\ O & O & O & \dots & \Sigma & U \end{pmatrix}_{2n\times 2n} \tag{4.113}$$

where $m = 2n, U = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$, $\beta \neq 0$, $\Sigma = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ and O is 2×2 zero matrix.

In this case, we use the following transformation

$$\begin{aligned}
\phi_{11} &= \tilde{\phi}_{11}, \\
\phi_{12} &= \tilde{\phi}_{12}, \\
\phi_{21} &= \tilde{\phi}_{21} + \frac{\partial}{\partial \beta}\tilde{\phi}_{11}, \\
\phi_{22} &= \tilde{\phi}_{22} + \frac{\partial}{\partial \beta}\tilde{\phi}_{12}, \\
&\vdots \qquad\qquad \vdots \\
\phi_{n1} &= \tilde{\phi}_{n1} + \frac{\partial}{\partial \beta}\tilde{\phi}_{n-1,1} + \cdots + \frac{1}{(n-1)!}\frac{\partial^{n-1}}{\partial \beta^{n-1}}\tilde{\phi}_{11}, \\
\phi_{n2} &= \tilde{\phi}_{n2} + \frac{\partial}{\partial \beta}\tilde{\phi}_{n-1,2} + \cdots + \frac{1}{(n-1)!}\frac{\partial^{n-1}}{\partial \beta^{n-1}}\tilde{\phi}_{12},
\end{aligned}$$

for obtaining systems (4.89)-(4.92).

Acknowledgement

The authors would like to thank Professor Pham Loi Vu for his valuable suggestions and comments.

References

[1] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, Studies in Applied Mathematics, SIAM, Philadelphia, 1981.

[2] S. F. Deng, D. Y. Chen and D. J. Zhang, The multisoliton solutions of the KP equation with self-consistent sources, *J. Phys. Soc. Jpn.* **72** (2003), 2184-2192.

[3] R. Hirota, *Direct Method in Soliton Theory*, (Edited and translated by A. Nagai, J. Nimmo and C. Gilson), Cambridge Univ. Press, Cambridge, 2004.

[4] J. J. C. Nimmo and N. C. Freeman, The use of Bäcklund transformations in obtaining N-soliton solutions in Wronskian form, *J. Phys. A: Math. Gen.* **17** (1984), 1415-1424.

[5] J. J. C. Nimmo, A method of obtaining the N-soliton solution of the Boussinesq equation in terms of Wronskian, *Phys. Lett.* **95** (1983), 4-6.

[6] J. J. C. Nimmo, A bilinear Bäcklund transformation for the nonlinear Schrödinger equation, *Phys. Lett.* **99** (1983), 279-280.

[7] T.K. Ning, D. J. Zhang, D. Y. Chen and S. F. Deng, Exact solutions and conservation laws for a nonisospectral sine-Gordon equation, *Chaos, Solitons and Fractals* **25** (2005), 611-620.

[8] J. Satsuma, A Wronskian representation of N-soliton solutions of nonlinear evolution equations, *J. Phys. Soc. Jpn.* **46** (1979), 359-360.

[9] W. X. Ma, Wronskians, generalized Wronskians and solutions to the Korteweg-de Vries equation, *Chao, Soliton and Fractals* **19** (2004), 163-170.

[10] W. X. Ma, Complexiton solutions of integrable equations, *Nonlinear Analysis* **63** (2005), e2461-e2471.

[11] W. X. Ma and Y. You, Solving the Korteweg-de Vries equation by its bilinear form: Wronskian solutions, *Trans. Amer. Math. Soc.* **357** (2005), 1753-1778.

[12] Pham Loi Vu, Explicit complex-valued solutions of the Korteweg-deVries equation on the half-line and on the whole-line, *Acta Applicandae Mathematicae* **49** (1997), 107-149.

[13] Pham Loi Vu and Nguyen Huy Hoang, On the degree of nomalization polynomials of the scattering data for constructing solutions of the Korteweg-de Vries equation, *Southeast Asian Bulletin of Mathematics* **24** (2000), 631-641.

[14] Pham Loi Vu and Nguyen Huy Hoang, Constructing Soliton Solutions of the Nonlinear Schrodinger Equation by Inverse Scattering and Hirota′s Direct Methods, *Vietnam J. Math.* **30** (2003), 149-165.

[15] V.E. Zakhazov, S. V. Manakov, S. P. Novikov and I. P. Pitaievski, *Theory of Solitons. The Inverse Scattering Method* (in Russian) Nauka, Moscow 1980 (English transl., Plenum, New York, 1984).

[16] D. J. Zhang, The N-soliton solutions for the modified KdV equation with self-consistent sources, *J. Phys. Soc. Jpn.* **71** (2002), 2649-2656.

[17] D. J. Zhang and D. Y. Chen , The N-soliton solutions of the sine-Gordon equation with self-consistent sources, *Physica A* **321** (2002), 467-481.

[18] D. J. Zhang, The N-soliton solutions of some soliton equations with self-consistent sources, *Chaos, Solitons and Fractals* **18** (2003), 31-43.

[19] D. J. Zhang, Notes on solutions in Wronskian form to soliton equation: KdV-type, preprint, 2006, arXiv:nlin. SI/0603008.

[20] Y. Zhang, S. F. Deng, D. J. Zhang and D. Y. Chen, The N-soliton for the non-isospectral mKdV equation, *Physica A* **339** (2004), 228-236.

Institute of Mathematics,
18 Hoang Quoc Viet Road,
Cau Giay district, Hanoi, Vietnam.
E-mail address: htngoan@math.ac.vn

University of Transport and Communications,
Lang Thuong Street, Dong Da district, Hanoi, Vietnam.
E-mail address: hoangnh@uct.edu.vn