ACTA MATHEMATICA VIETNAMICA
Volume 36, Number 2, 2011, pp. 145–167

SOME PROPERTIES OF ORLICZ-LORENTZ SPACES

HA HUY BANG, NGUYEN VAN HOANG AND VU NHAT HUY

Dedicated to Tran Duc Van on the occasion of his sixtieth birthday

Abstract. In this paper we study some fundamental properties of Orlicz-Lorentz spaces defined on $\mathbb{R}$ such as finding their dual spaces, giving best constants for the inequalities between the Orlicz norm and the Luxemburg norm on Orlicz-Lorentz spaces and establishing the Kolmogorov inequality in these spaces.

1. Orlicz-Lorentz spaces

Orlicz-Lorentz spaces as a generalization of Orlicz spaces L_φ and Lorentz spaces Λ_ω have been studied by many authors (we refer to [9, 10, 11, 12, 14, 18, 19] for basic properties of Orlicz- Lorentz spaces as well to the references therein). In this paper we study some fundamental properties of Orlicz-Lorentz spaces defined on the real line $\Lambda^{\mathbb{R}}_{\varphi,\omega}$. We first find the dual spaces of $\Lambda^{\mathbb{R}}_{\varphi,\omega}$. Note that the dual spaces of Orlicz-Lorentz spaces defined on $(0,+\infty)$ or $(0,1)$ were studied in [11]. Next we introduce the Orlicz norm on $\Lambda^{\mathbb{R}}_{\varphi,\omega}$ which defined by using the $M^{\mathbb{R}}_{\varphi_*,\omega}$ space and then we give a simple formula to calculate the Orlicz norm directly by φ,ω. On Orlicz spaces, it is known that the Orlicz norm and the Luxemburg norm are equivalent, and it will be shown that the corresponding norms on Orlicz-Lorentz spaces are also equivalent. Moreover, we investigate best constants C_1, C_2 for the inequalities between the Orlicz norm and the Luxemburg norm on Orlicz-Lorentz spaces and we notice that these results for the special case when $\Lambda^{\mathbb{R}}_{\varphi,\omega}$ becomes Orlicz spaces will be published in [2]. The dual equality between the Orlicz norm on Orlicz-Lorentz spaces and the norm on $M^{\mathbb{R}}_{\varphi_*,\omega}$ is also given. Finally, we prove the Kolmogorov inequality in the Orlicz-Lorentz spaces.

Let us first recall some notations of Orlicz- Lorentz spaces:
Let $(\Omega,\mu) := (\Omega,\Sigma,\mu)$ be a measure space with the complete and σ-finite measure μ, $L^0(\mu)$ be a space of all μ-equivalent classes of Σ-measurable functions on Ω with topology of the convergence in measure on μ-finite sets.
A Banach space $(E, \|.\|_E)$ is called the Banach function space on (Ω,μ) if it is a subspace of $L^0(\mu)$, and there exists a function $h \in E$ such that $h > 0$ a.e. on

Received May 9, 2011.
2000 *Mathematics Subject Classification.* 46E30.
Key words and phrases. Orlicz- Lorentz spaces.
This work was supported by the Vietnam National Foundation for Science and Technology Development (Project N0 101.01.50.09).

Ω and if $f \in L^0(\mu)$, $g \in E$ and $|f| \leq |g|$ a.e. on Ω then $f \in E$ and we have $\|f\|_E \leq \|g\|_E$. Moreover, if the unit ball $B_E = \{f \in E : \quad \|f\|_E \leq 1\}$ is closed on $L^0(\mu)$, then we say that E has the Fatou property. A Banach function space E is said to be symmetric if for every $f \in L^0(\mu)$ and $g \in E$ such that $\mu_f = \mu_g$, then $f \in E$ and $\|f\|_E = \|g\|_E$, where for any $h \in L^0(\mu)$, μ_h denotes the distribution of h, defined by

$$\mu_h(t) = \mu(\{x \in \Omega : \quad |h(x)| > t\}), \quad t \geq 0.$$

Let E be a Banach function space on (Ω, μ). Then the Köthe dual space $E^{'}$ of E is a Banach function space, which can be identified with the space of all functionals possessing an integral representation, that is,

$$E^{'} = \{g \in L^0(\mu) : \quad \|g\|_{E'} = \sup_{\|f\|_E \leq 1} \int_\Omega |fg| d\mu < \infty\}.$$

Given $\varphi : [0, \infty) \to [0, \infty)$ an Orlicz function (i.e., it is a convex function, takes value zero only at zero) and $\omega : (0, \infty) \to (0, \infty)$ a weight function (i.e., it is a non-increasing function and locally integrable and $\int_0^\infty \omega dx = \infty$). The Orlicz - Lorentz space $\Lambda^\Omega_{\varphi,\omega}$ on (Ω, μ) is the set of all functions $f(x) \in L^0(\mu)$ such that

$$\int_0^\infty \varphi(\lambda f^*(x))\omega(x) dx < \infty$$

for some $\lambda > 0$, where f^* is the non-increasing rearrangement of f defined by

$$f^*(x) = \inf\{\lambda > 0 : \quad \mu_f(\lambda) \leq x\},$$

with $x > 0$ (by convention, $\inf \emptyset = \infty$).
It is easy to check that $\Lambda^\Omega_{\varphi,\omega}$ is a symmetric Banach function space, with the Fatou property, equipped with the Luxemburg norm

$$\|f\|_{\Lambda^\Omega_{\varphi,\omega}} = \inf\{\lambda > 0 : \quad \int_0^\infty \varphi(\frac{f^*(x)}{\lambda})\omega(x) dx \leq 1\}.$$

Note that: If $\omega \equiv 1$ then $\Lambda^\Omega_{\varphi,\omega}$ is the Orlicz function space L^Ω_φ; if $\varphi(t) = t$ then $\Lambda^\Omega_{\varphi,\omega}$ is the Lorentz function space Λ^Ω_ω.
Recall that φ is an N-function if $\lim\limits_{t \to 0} \varphi(t)/t = 0$ and $\lim\limits_{t \to +\infty} \varphi(t)/t = +\infty$; the Orlicz function φ satisfies Δ_2-condition (we write, $\varphi \in \Delta_2$) if there exists $C > 0$ such that $\varphi(2t) \leq C\varphi(t) \quad \forall t > 0$; the Orlicz function $\varphi : [0, +\infty) \longmapsto [0, +\infty)$ satisfies the ∇_2-condition (we write, $\varphi \in \nabla_2$) if there exists a number $l > 1$ such that $\varphi(x) \leq \frac{1}{2l}\varphi(lx) \quad \forall x \geq 0$. We easily have the following remarks:

Remark 1.1.
(i) If $f(x) \in E^{'}$ and $0 \leq f_n \uparrow |f|$ then $\lim\limits_{n \to +\infty} \|f_n\|_{E'} = \|f\|_{E'}$;
(ii) If $f(x), f_n(x), n = 1, 2, ...$ are measurable functions satisfying $|f_n| \uparrow |f|$ then $f^*_n \uparrow f^*$;

(iii) If $f(x), g(x)$ are measurable functions then

$$\int\limits_{\Omega} |f(x)g(x)|d\mu \leq \int\limits_{0}^{+\infty} f^*(x)g^*(x)dx.$$

Remark 1.2. Let φ be an N-function. Then the three following conditions are equivalent:
(i) $\varphi \in \nabla_2$;
(ii) There exists $\beta > 1$ such that $x\psi(x) > \beta\varphi(x) \quad \forall x > 0$, where $\psi(x)$ is the left derivative of φ;
(iii) There are the numbers $l > 1$ and $\delta_l > 0$ such that $\varphi(lx) \geq (l+\delta_l)\varphi(x) \forall x > 0$.

Denote by φ_* the Young conjugate function of φ, that is

$$\varphi_*(t) = \sup\{st - \varphi(s) | s \geq 0\}, \quad t \geq 0.$$

Then we have the following result
Young's inequality. *Let φ be an N-function. Then*

$$xy \leq \varphi(x) + \varphi_*(y) \quad \forall x, y \geq 0$$

and it becomes the equality if and only if $y \in [\psi(x), \eta(x)]$, where ψ, η are the left and the right derivatives of φ.
We define

$$I(f) = \int\limits_{0}^{\infty} \varphi_*\left(\frac{f^*(x)}{\omega(x)}\right)\omega(x)dx$$

for any $f(x) \in L^0(\mu)$ and

$$M^{\Omega}_{\varphi_*,\omega} = \{f(x) \in L^0(\mu) : \quad I\left(\frac{f}{\lambda}\right) < \infty \quad \text{with some } \lambda > 0\}.$$

In the space $M^{\Omega}_{\varphi_*,\omega}$ we define a monotone and homogeneous functional

$$\|f\|_{M^{\Omega}_{\varphi_*,\omega}} = \inf\{\lambda > 0 : \quad I\left(\frac{f}{\lambda}\right) \leq 1\}.$$

Put

$$S(t) = \int\limits_{0}^{t} \omega(s)ds, \quad t > 0,$$

we call the weight function ω regular if there is a constant $K > 1$ such that $S(2t) \geq KS(t)$ for any $t > 0$. It is easy to prove that ω is regular if and only if there exists $C > 0$ such that $t\omega(t) \leq S(t) \leq Ct\omega(t)$ for any $t > 0$.
Let $f(x), g(x)$ be two positive functions, we write $f \asymp g$ if there exist $C_1, C_2 > 0$ such that $C_1 f(x) \leq g(x) \leq C_2 f(x)$. Put $I = (0, +\infty)$.
It was proved in [11] that
Theorem A. *Let ω be a weight function and $\varphi(t) = t$ or φ be an N-function. Then the following assertions are true:*
(i) *If ω is regular, then $(\Lambda^I_{\varphi,\omega})' = M^I_{\varphi_*,\omega}$ and $\|.\|_{(\Lambda^I_{\varphi,\omega})'} \asymp \|.\|_{M^I_{\varphi_*,\omega}}$;*

(ii) *If* $\varphi \in \Delta_2$ *and* $(\Lambda^I_{\varphi,\omega})' = M^I_{\varphi_*,\omega}$, *then* ω *is regular.*

Theorem A shows the relation between Orlicz-Lorentz spaces and $M^I_{\varphi_*,\omega}$, that is, the Köthe dual space of Orlicz-Lorentz space $(\Lambda^I_{\varphi,\omega})$ is the $M^I_{\varphi_*,\omega}$ with some conditions of φ, ω.

2. Main results

We state the following theorem as an extension of Theorem A

Theorem 2.1. *Let* ω *be a weight function and* $\varphi(t) = t$ *or* φ *be an N-function. Then the following assertions are true*

(i) *If* ω *is regular, then* $(\Lambda^{\mathbb{R}}_{\varphi,\omega})' = M^{\mathbb{R}}_{\varphi_*,\omega}$ *and* $\|.\|_{(\Lambda^{\mathbb{R}}_{\varphi,\omega})'} \asymp \|.\|_{M^{\mathbb{R}}_{\varphi_*,\omega}}$;

(ii) *If* $\varphi \in \Delta_2$ *and* $(\Lambda^{\mathbb{R}}_{\varphi,\omega})' = M^{\mathbb{R}}_{\varphi_*,\omega}$, *then* ω *is regular.*

To prove Theorem 2.1, we need the following lemmas.

Lemma 2.2. *Let* $\Omega = \mathbb{R}$ *or* I *and* $f(x)$ *be a measurable function. Then the following are true*

(i) $f \in \Lambda^{\Omega}_{\varphi,\omega}$ *if and only if* $f^* \in \Lambda^I_{\varphi,\omega}$, *and we have* $\|f\|_{\Lambda^{\Omega}_{\varphi,\omega}} = \|f^*\|_{\Lambda^I_{\varphi,\omega}}$;

(ii) $f \in M^{\Omega}_{\varphi_*,\omega}$ *if and only if* $f^* \in M^I_{\varphi_*,\omega}$, *and we have* $\|f\|_{M^{\Omega}_{\varphi_*,\omega}} = \|f^*\|_{M^I_{\varphi_*,\omega}}$;

(iii) $f \in (\Lambda^{\Omega}_{\varphi,\omega})'$ *if and only if* $f^* \in (\Lambda^I_{\varphi,\omega})'$, *and we have* $\|f\|_{(\Lambda^{\Omega}_{\varphi,\omega})'} = \|f^*\|_{(\Lambda^I_{\varphi,\omega})'}$.

Proof. (i) and (ii) is evident from their definitions. Let us prove (iii). Using (i) and Remark 1.1, we have if $f^*(x) \in (\Lambda^I_{\varphi,\omega})'$, then $f(x) \in (\Lambda^{\Omega}_{\varphi,\omega})'$ and

$$\|f\|_{(\Lambda^{\Omega}_{\varphi,\omega})'} \le \|f^*\|_{(\Lambda^I_{\varphi,\omega})'}.$$

Conversely, suppose that $f(x) \in (\Lambda^{\Omega}_{\varphi,\omega})'$, we have to prove $f^*(x) \in (\Lambda^I_{\varphi,\omega})'$ and $\|f^*\|_{(\Lambda^I_{\varphi,\omega})'} \le \|f\|_{(\Lambda^{\Omega}_{\varphi,\omega})'}$. By Remark 1.1, we only prove for $f(x)$ being a non-negative, simple function on Ω. So $f^*(x)$ is a nonnegative simple function on I, too. For any simple function $g(x) \in \Lambda^I_{\varphi,\omega}$ satisfying $\|g\|_{\Lambda^I_{\varphi,\omega}} \le 1$, there is a simple function $h(x)$ on Ω such that $h^*(x) = g^*(x)$ and

$$\int_{\Omega} |f(x)h(x)| d\mu = \int_{I} f^*(x) g^*(x) dx.$$

Hence, $h(x) \in \Lambda^{\Omega}_{\varphi,\omega}$ and $\|h\|_{\Lambda^{\Omega}_{\varphi,\omega}} \le 1$, and then

$$\int_{I} |f^*(x)g(x)| dx \le \int_{I} f^*(x)g^*(x) dx = \int_{\Omega} |f(x)h(x)| dx \le \|f\|_{(\Lambda^{\Omega}_{\varphi,\omega})'}.$$

If $g(x)$ is an arbitrary function in $\Lambda^I_{\varphi,\omega}$ such that $\|g\|_{\Lambda^I_{\varphi,\omega}} \le 1$, there is a sequence $\{g_n(x)\}$ of nonnegative simple functions on I such that $g_n(x) \uparrow |g(x)|$. So $g_n(x) \in \Lambda^I_{\varphi,\omega}$ and $\|g_n\|_{\Lambda^I_{\varphi,\omega}} \le 1$. By the monotone convergence theorem, we have

$$\int_{I} |f^*(x)g(x)| dx = \lim_{n\to\infty} \int_{I} |f^*(x)g_n(x)| dx \le \|f\|_{(\Lambda^{\Omega}_{\varphi,\omega})'}.$$

Hence, $f^*(x) \in (\Lambda^I_{\varphi,\omega})'$ and $\|f^*\|_{(\Lambda^I_{\varphi,\omega})'} \leq \|f\|_{(\Lambda^\Omega_{\varphi,\omega})'}$. The proof is complete. □

Lemma 2.3. *Let φ be an Orlicz function, ω be a weight function and suppose that $(\Lambda^{\mathbb{R}}_{\varphi,\omega})' = M^{\mathbb{R}}_{\varphi_*,\omega}$. Then there is a constant $K > 0$ satisfying*

$$\|g\|_{M^{\mathbb{R}}_{\varphi_*,\omega}} \leq K\|g\|_{(\Lambda^{\mathbb{R}}_{\varphi,\omega})'} \quad \forall g(x) \in (\Lambda^{\mathbb{R}}_{\varphi,\omega})'.$$

Proof. This lemma is proved similarly as in the proof of Lemma 2 in [11]. □

Proof of Theorem 2.1. (i). By the regularity of ω, we have $(\Lambda^I_{\varphi,\omega})' = M^I_{\varphi_*,\omega}$ and

$$2^{-1}\|f\|_{(\Lambda^I_{\varphi,\omega})'} \leq \|f\|_{M^I_{\varphi_*,\omega}} \leq 4C\|f\|_{(\Lambda^I_{\varphi,\omega})'}$$

where C is a constant (see Theorem 2 (i) [11]). It follows from Lemma 2.2 that $f(x) \in (\Lambda^{\mathbb{R}}_{\varphi,\omega})' \iff f^* \in (\Lambda^I_{\varphi,\omega})' = M^I_{\varphi_*,\omega} \iff f(x) \in M^{\mathbb{R}}_{\varphi_*,\omega}$. Therefore, $(\Lambda^{\mathbb{R}}_{\varphi,\omega})' = M^{\mathbb{R}}_{\varphi_*,\omega}$, and

$$\|f\|_{(\Lambda^{\mathbb{R}}_{\varphi,\omega})'} = \|f^*\|_{(\Lambda^I_{\varphi,\omega})'} \leq 2\|f^*\|_{M^I_{\varphi_*,\omega}} = 2\|f\|_{M^{\mathbb{R}}_{\varphi_*,\omega}},$$

$$\|f\|_{M^{\mathbb{R}}_{\varphi_*,\omega}} = \|f^*\|_{M^I_{\varphi_*,\omega}} \leq 4C\|f^*\|_{(\Lambda^I_{\varphi,\omega})'} = 4C\|f\|_{(\Lambda^{\mathbb{R}}_{\varphi,\omega})'}.$$

i.e., $\|.\|_{(\Lambda^{\mathbb{R}}_{\varphi,\omega})'} \asymp \|.\|_{M^{\mathbb{R}}_{\varphi_*,\omega}}$.

By Theorem 2(ii) [11], we will prove that $(\Lambda^I_{\varphi,\omega})' = M^I_{\varphi_*,\omega}$. From Young's inequality we have $M^I_{\varphi_*,\omega} \subset (\Lambda^I_{\varphi,\omega})'$, so we only have to prove $(\Lambda^I_{\varphi,\omega})' \subset M^I_{\varphi_*,\omega}$. Given $f \in (\Lambda^I_{\varphi,\omega})'$, there is a sequence $\{f_n\}$ of nonnegative simple functions on I such that $f_n \uparrow |f|$. Hence, $f_n \in (\Lambda^I_{\varphi,\omega})'$ for all $n \in \mathbb{N}$ and $\|f_n\|_{(\Lambda^I_{\varphi,\omega})'} \uparrow \|f\|_{(\Lambda^I_{\varphi,\omega})'}$. We choose a sequence $\{h_n\}$ of simple functions on $\mathbb{R}$ such that $h^*_n = f^*_n$ for all $n \in \mathbb{N}$. Thus $h_n \in (\Lambda^{\mathbb{R}}_{\varphi,\omega})'$ and $\|h_n\|_{(\Lambda^{\mathbb{R}}_{\varphi,\omega})'} = \|f_n\|_{(\Lambda^I_{\varphi,\omega})'}$. By assumption $(\Lambda^{\mathbb{R}}_{\varphi,\omega})' = M^{\mathbb{R}}_{\varphi_*,\omega}$, we obtain $h_n \in M^{\mathbb{R}}_{\varphi_*,\omega}$. Lemma 2.2(ii) yields $f^*_n = h^*_n \in M^I_{\varphi_*,\omega}$ and $\|f^*_n\|_{M^I_{\varphi_*,\omega}} = \|h^*_n\|_{M^I_{\varphi_*,\omega}} = \|h_n\|_{M^{\mathbb{R}}_{\varphi_*,\omega}}$. From Lemma 2.3, there is a constant $K > 0$ such that

$$\|g\|_{M^{\mathbb{R}}_{\varphi_*,\omega}} \leq K\|g\|_{(\Lambda^{\mathbb{R}}_{\varphi,\omega})'} \quad \forall g \in (\Lambda^{\mathbb{R}}_{\varphi,\omega})'.$$

Hence

$$\|f^*_n\|_{M^I_{\varphi_*,\omega}} = \|h_n\|_{M^{\mathbb{R}}_{\varphi_*,\omega}} \leq K\|h_n\|_{(\Lambda^{\mathbb{R}}_{\varphi,\omega})'} = K\|f_n\|_{(\Lambda^I_{\varphi,\omega})'} \leq K\|f\|_{(\Lambda^I_{\varphi,\omega})'}.$$

So $\sup\limits_{n\geq 1}\|f^*_n\|_{M^I_{\varphi_*,\omega}} < \infty$. Then it follows from $f^*_n \uparrow f^*$ that $f^* \in M^I_{\varphi_*,\omega}$. Therefore, $f \in M^I_{\varphi_*,\omega}$. That is $(\Lambda^I_{\varphi,\omega})' \subset M^I_{\varphi_*,\omega}$. The proof is complete. □

Next, we study the norms on Orlicz-Lorentz spaces. We know that $\Lambda^\Omega_{\varphi,\omega}$ is the Banach space with the Luxemburg norm defined by

$$\|f\|_{\Lambda^\Omega_{\varphi,\omega}} = \inf\{\lambda > 0 : \quad \int\limits_I \varphi(\frac{f^*(x)}{\lambda})\omega(x)dx \leq 1\}.$$

On this space we define another norm called the Orlicz norm as follows

$$\|f\|^1_{\Lambda^{\Omega}_{\varphi,\omega}} = \sup\{\int\limits_{\Omega} |f(x)g(x)|d\mu : \quad \|g\|_{M^{\Omega}_{\varphi_*,\omega}} \leq 1\}. \tag{1}$$

Note that if $\omega \equiv 1$ then $\Lambda^{\Omega}_{\varphi,\omega}$ is the usual Orlicz function space L^{Ω}_{φ} and the Luxemburg norm, the Orlicz norm on $\Lambda^{\Omega}_{\varphi,\omega}$ become respectively the Luxemburg norm, the Orlicz norm on L^{Ω}_{φ} which can be seen in [19].
From the definition of the Orlicz norm, we have

$$\int\limits_{\Omega} |f(x)g(x)|d\mu \leq \|f\|^1_{\Lambda^{\Omega}_{\varphi,\omega}} \|g\|_{M^{\Omega}_{\varphi_*,\omega}} \quad \forall f(x) \in \Lambda^{\Omega}_{\varphi,\omega}, g(x) \in M^{\Omega}_{\varphi_*,\omega}.$$

In the following theorem, we give a formula to compute the Orlicz norm.

Theorem 2.4. *Let φ be an N-function and $f(x) \in \Lambda^{\mathbb{R}}_{\varphi,\omega}$. Then*

$$\|f\|^1_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} = \inf\{\frac{1}{k}(1 + \int\limits_{I} \varphi(kf^*(x))\omega(x)dx) : \quad k > 0\}. \tag{2}$$

To obtain Theorem 2.4, the following result is important.

Lemma 2.5. *Let φ be an Orlicz function, ψ be the left derivative of φ and the function $f(x) \in \Lambda^{\mathbb{R}}_{\varphi,\omega}$ satisfy*

$$\int\limits_{I} \varphi_*(\psi(k_0 f^*(x))\omega(x)dx = 1 \tag{3}$$

for some k_0. Then we have

$$\|f\|^1_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} = \int\limits_{I} f^*(x)\psi(k_0 f^*(x))\omega(x)dx = \frac{1}{k_0}(1 + \int\limits_{I} \varphi(k_0 f^*(x))\omega(x)dx).$$

Proof. From (1) and (3), we have

$$\begin{aligned} \|f\|^1_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} &= \sup\{\int\limits_{\mathbb{R}} |f(x)g(x)|dx : \quad \|g\|_{M^{\mathbb{R}}_{\varphi_*,\omega}} \leq 1\} \\ &\leq \frac{1}{k_0}(1 + \int\limits_{I} \varphi(k_0 f^*(x))\omega(x)dx) \\ &= \frac{1}{k_0}(\int\limits_{I} \varphi_*(\psi(k_0 f^*(x))\omega(x)dx + \int\limits_{I} \varphi(k_0 f^*(x))\omega(x)dx) \\ &= \int\limits_{I} f^*(x)\psi(k_0 f^*(x))\omega(x)dx. \end{aligned} \tag{4}$$

We define $g(x) = \omega(x)\psi(k_0 f^*(x)), x > 0$. Then $\|g\|_{M^I_{\varphi_*,\omega}} \leq 1$. Therefore, it follows from the definition of the Orlicz norm that

$$\|f\|^1_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} = \|f^*\|^1_{\Lambda^I_{\varphi,\omega}} \geq \int_I f^*(x)g(x)dx = \int_I f^*(x)\psi(k_0 f^*(x))\omega(x)dx. \tag{5}$$

From (4) and (5), the proof is complete. □

Proof of Theorem 2.4. For $k > 0$ and $g(x) \in M^{\mathbb{R}}_{\varphi_*,\omega}$ satisfying $\|g\|_{M^{\mathbb{R}}_{\varphi_*,\omega}} \leq 1$, we have

$$\int_I \varphi_*(\frac{g^*(x)}{\omega(x)})\omega(x)dx \leq 1.$$

Thus

$$\begin{aligned}\int_{\mathbb{R}} |f(x)g(x)|dx &\leq \int_I f^*(x)g^*(x)dx = \frac{1}{k}\int_I (\frac{g^*(x)}{\omega(x)})(kf^*(x))\omega(x)dx \\ &\leq \frac{1}{k}(\int_I \varphi_*(\frac{g^*(x)}{\omega(x)})\omega(x)dx + \int_I \varphi(kf^*(x))\omega(x)dx) \\ &\leq \frac{1}{k}(1 + \int_I \varphi(kf^*(x))\omega(x)dx).\end{aligned}$$

This inequality implies that

$$\|f\|^1_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} \leq \inf\{\frac{1}{k}(1 + \int_I \varphi(kf^*(x))\omega(x)dx) : \ k > 0\}. \tag{6}$$

Next, we will prove the opposite inequality: Let us first suppose that ψ is a continuous function. There exists a sequence $\{f_n(x)\}$ of nonnegative, simple functions such that $f_n(x) \uparrow |f(x)|$ a.e. Hence, $\|f_n\|^1_{\Lambda_{\varphi,\omega}} \uparrow \|f\|^1_{\Lambda_{\varphi,\omega}}$, and $f^*_n(x) \uparrow f^*(x)$. We have $m(\{x \in \mathbb{R} : f_n(x) \neq 0\}) < \infty$ for all $n \in \mathbb{N}$ and φ_* is a continuous function. So there is a sequence $\{k_n\}$ of positive numbers such that

$$\int_I \varphi_*(\psi(k_n f^*_n(x)))\omega(x)dx = 1 \quad \forall n \in \mathbb{N}.$$

Therefore, it follows from Lemma 2.5 that

$$k_n\|f_n\|^1_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} = \int_I \varphi(k_n f^*(x))\omega(x)dx + 1 \quad \forall n \in \mathbb{N}. \tag{7}$$

Because $\{f^*_n(x)\}$ is an increasing sequence of functions and $\{k_n\}$ is a decreasing sequence, there exists the limit $\lim_{n\to\infty} k_n = k_*$, and then from (7), we have

$$k_*\|f\|^1_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} = \liminf_{n\to\infty} \int_I \varphi(k_n f^*(x))\omega(x)dx + 1 \geq \int_I \varphi(k_* f^*(x))\omega(x)dx + 1.$$

Therefore, $k_* > 0$ and

$$\|f\|^1_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} \geq \frac{1}{k_*}(\int_I \varphi(k_* f^*(x))\omega(x)dx + 1)$$
$$\geq \inf\{\frac{1}{k}(1 + \int_I \varphi(kf^*(x))\omega(x)dx) : \ k > 0\}.$$

From this and (6), we get

$$\|f\|^1_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} = \inf\{\frac{1}{k}(1 + \int_I \varphi(kf^*(x))\omega(x)dx) : \ k > 0\}.$$

If ψ is not a continuous function, it is known that ψ is left continuous. For all $\epsilon \in (0, 1)$, we approximate ψ by a nondecreasing, continuous ψ_1 such that

$$\varphi((1-\epsilon)x) \leq \varphi_1(x) \leq \varphi(x) \quad \forall x > 0,$$

where φ_1 is a convex function having ψ_1 as the left derivative. It is easy to prove that φ_1 is an N-function, $\Lambda^{\mathbb{R}}_{\varphi,\omega} = \Lambda^{\mathbb{R}}_{\varphi_1,\omega}$ and

$$(1-\epsilon)\|f\|^1_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} \leq \|f\|^1_{\Lambda^{\mathbb{R}}_{\varphi_1,\omega}} \leq \|f\|^1_{\Lambda^{\mathbb{R}}_{\varphi,\omega}}.$$

Hence, due to the result proved for continuous functions , we have

$$\|f\|^1_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} \geq \|f\|^1_{\Lambda^{\mathbb{R}}_{\varphi_1,\omega}} = \inf\{\frac{1}{k}(1 + \int_I \varphi_1(kf^*(x))\omega(x)dx : \ k > 0\}$$
$$\geq \inf\{\frac{1}{k}(1 + \int_I \varphi((1-\epsilon)kf^*(x))\omega(x)dx : \ k > 0\}$$
$$= (1-\epsilon)\inf\{\frac{1}{k}(1 + \int_I \varphi(kf^*(x))\omega(x)dx : \ k > 0\}.$$

Letting $\epsilon \to 0$, we get

$$\|f\|^1_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} \geq \inf\{\frac{1}{k}(1 + \int_I \varphi(kf^*(x))\omega(x)dx : \ k > 0\}.$$

Combining this equation with (6) we obtain

$$\|f\|^1_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} = \inf\{\frac{1}{k}(1 + \int_I \varphi(kf^*(x))\omega(x)dx : \ k > 0\}.$$

The proof is complete. □

We will show, in particular, that on Orlicz-Lorentz spaces the Orlicz norm and the Luxemburg norm are equivalent , and satisfy the following inequalities

$$\|f\|_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} \leq \|f\|^1_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} \leq 2\|f\|_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} \quad \forall f \in \Lambda^{\mathbb{R}}_{\varphi,\omega}.$$

Moreover, we find the best constants for the inequalities between these norms. Suppose that C_1 is the largest number and C_2 is the smallest number such that

$$C_1\|f\|_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} \le \|f\|^1_{\Lambda^{|R}_{\varphi,\omega}} \le C_2\|f\|_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} \quad \forall f \in \Lambda^{\mathbb{R}}_{\varphi,\omega}. \tag{8}$$

It is well known that the Orlicz norm has the Fatou property, that is, if $0 \le f_n \le f \in \Lambda^{\mathbb{R}}_{\varphi,\omega}$, then $\|f_n\|_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} \to \|f\|_{\Lambda^{\mathbb{R}}_{\varphi,\omega}}$ whenever $f_n \to f$ a.e. Hence, condition (8) is equivalent to the following

$$C_1\|f\|_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} \le \|f\|^1_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} \le C_2\|f\|_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} \tag{9}$$

for all functions $f(x) \in \Lambda^{\mathbb{R}}_{\varphi,\omega}$ which are nonnegative, simple and satisfying $\|f\|_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} = 1$. From the above definition, we have $1 \le C_1 \le C_2 \le 2$. Put

$$H(k) = \sup_{t>0} \frac{\varphi(kt)}{\varphi(t)}, \qquad D(k) = \inf_{t>0} \frac{\varphi(kt)}{\varphi(t)}.$$

Clearly, the functions $D(k), H(k)$ are increasing, $D(k) \le H(k) \le k$ for any $0 \le k \le 1$ and $k \le D(k) \le H(k)$ for any $k > 1$. From now on, we denote by f^{-1} the inverse function of f.

Theorem 2.6. *Let φ be an Orlicz function. Then*

(i) *If φ is an N-function, we have*

$$C_1 = \inf_{c>0} \frac{1}{c}\varphi_*^{-1}(c)\varphi^{-1}(c) = \inf_{k>0} \frac{1+D(k)}{k}; \tag{10}$$

(ii) *We always have*

$$C_2 \le \inf_{k>0} \frac{1+H(k)}{k}, \tag{11}$$

and if φ is an N-function then

$$\sup_{c>0} \frac{1}{c}\varphi_*^{-1}(c)\varphi^{-1}(c) \le C_2. \tag{12}$$

Proof. (i) We have

$$\begin{aligned} \inf_{k>0} \frac{1}{k}(1+D(k)) &= \inf_{k>0} \frac{1}{k}\left(1+\inf_{x>0}\frac{\varphi(kx)}{\varphi(x)}\right) = \inf_{k>0}\frac{1}{k}(1+\inf_{c>0}\frac{\varphi(k\varphi^{-1}(c))}{c}) \\ &= \inf_{c>0}\frac{1}{c}\inf_{k>0}\frac{\varphi_*(\varphi_*^{-1}(c)+\varphi(k\varphi^{-1}(c))}{kc} = \inf_{c>0}\frac{1}{c}\varphi^{-1}(c)\varphi_*^{-1}(c). \end{aligned} \tag{13}$$

If $f(x) \in \Lambda^{\mathbb{R}}_{\varphi,\omega}$ is a simple function such that $\|f\|_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} = 1$, then

$$\int_I \varphi(f^*(x))\omega(x)dx = 1.$$

Therefore, it follows from Theorem 2.4 that

$$\begin{aligned}\|f\|^1_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} &= \inf\Big\{\frac{1}{k}\Big(1+\int_I \varphi(kf^*(x))\omega(x)\,dx\Big):\ k>0\Big\}\\ &\geq \inf\Big\{\frac{1}{k}\Big(1+D(k)\int_I \varphi(f^*(x))\omega(x)\,dx\Big):\ k>0\Big\}\geq \inf_{k>0}\frac{1+D(k)}{k}.\end{aligned}$$

Therefore, due to (9), we have

$$C_1 \geq \inf_{k>0}\frac{1+D(k)}{k}. \tag{14}$$

For any $c>0$, we choose $t>0$ such that $\int_0^t \omega(x)dx = 1/c$. We put $f(x)=\chi_{(0,t)}(x)$. By an immediate computation, we have

$$\|f\|_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} = \frac{1}{\varphi^{-1}(c)} \quad \text{and} \quad \|f\|^1_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} = \inf_{k>0}\frac{1}{k}(1+\varphi(k)c) = \frac{1}{c}\varphi_*^{-1}(c).$$

Hence

$$C_1 \leq \frac{\|f\|^1_{\Lambda^{\mathbb{R}}_{\varphi,\omega}}}{\|f\|_{\Lambda^{\mathbb{R}}_{\varphi,\omega}}} = \frac{1}{c}\varphi_*^{-1}(c)\varphi^{-1}(c) \quad \forall c>0.$$

This implies

$$C_1 \leq \inf_{c>0}\frac{1}{c}\varphi^{-1}(c)\varphi_*^{-1}(c). \tag{15}$$

Combining (13), (14) and (15), we obtain (10).
(ii) Let $f\in\Lambda^{\mathbb{R}}_{\varphi,\omega}$ and $\|f\|^1_{\Lambda^{\mathbb{R}}_{\varphi,\omega}}\leq 1$. Then it follows from Theorem 2.4 that

$$\begin{aligned}\|f\|^1_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} &\leq \frac{1}{k}(1+\int_I \varphi(kf^*(x))\omega(x)dx)\\ &\leq \frac{1}{k}(1+H(k)\int_I \varphi(f^*(x))\omega(x)dx) \leq \frac{1+H(k)}{k} \quad \forall k>0,\end{aligned}$$

which gives $C_2 \leq \inf_{k>0}\frac{1+H(k)}{k}$.
Similarly as above for C_1, we get

$$C_2 \geq \sup_{c>0}\frac{1}{c}\varphi^{-1}(c)\varphi_*^{-1}(c),$$

if φ is an N-function. The proof is complete. □

Now we find the conditions so that $C_1=1$.

Theorem 2.7. *Let φ be an N-function. Then $C_1>1$ if and only if $\varphi\in\Delta_2\cap\nabla_2$.*

Proof. Necessity. Assume $C_1 > 1$. We have to prove that $\varphi \in \Delta_2 \cap \nabla_2$. Indeed, assume the contrary that $\varphi \notin \Delta_2 \cap \nabla_2$. Then $\varphi \notin \Delta_2$ or $\varphi \notin \nabla_2$. From Theorem 2.6, we have

$$C_1 = \inf_{t>0} \frac{1 + D(t)}{t}.$$

If $\varphi \notin \Delta_2$, there exists a sequence of positive numbers $\{x_n\}$ such that $\varphi(x_n) \geq n\varphi(x_n/2)$ $\forall n \in \mathbb{N}$. Fix $t \in (0, 1)$ and choose $n_0 \in \mathbb{N}$ such that $1/2 \geq t^{n_0}$. Then for all $n > n_0$ we have $\varphi(x_n) \geq n\varphi(x_n/2) \geq n\varphi(t^{n_0}x_n)$. So, it follows from $\varphi(t^{n_0}x_n) \geq (D(t))^{n_0}\varphi(x_n)$ that $1 \geq n(D(t))^{n_0}$ $\quad \forall n > n_0$, and then $D(t) = 0$ for all $t \in (0, 1)$. Hence

$$C_1 \leq \inf_{t\in(0,1)} \frac{1 + D(t)}{t} = \inf_{t\in(0,1)} \frac{1}{t} = 1.$$

Therefore, by $C_1 \geq 1$ we have $C_1 = 1$.
If $\varphi \notin \nabla_2$, it follows from Remark 1.2 that for any $t > 1$, for all $\delta > 0$ there exists $x > 0$ such that

$$\varphi(tx) < (t + \delta)\varphi(x).$$

Therefore,

$$D(t) = \inf_{x>0} \frac{\varphi(tx)}{\varphi(x)} \leq t + \delta.$$

Letting $\delta \to 0$, we obtain $D(t) = t$ $\quad \forall t > 1$. So we have

$$C_1 \leq \inf_{t>1} \frac{1 + D(t)}{t} = \inf_{t>1} \frac{1 + t}{t} = 1.$$

From this inequality and by $C_1 \geq 1$, we get $C_1 = 1$, which contradicts $C_1 > 1$. So, $\varphi \in \Delta_2 \cap \nabla_2$ has been proved.
Sufficiency. Assume $\varphi \in \Delta_2 \cap \nabla_2$, we have to show $C_1 > 1$. Indeed, since $\varphi \in \Delta_2$, $D(1/2) > 0$. Since $\varphi \in \nabla_2$, there exists $\beta > 1$ such that

$$\frac{x\psi(x)}{\varphi(x)} > \beta \quad \forall x > 0,$$

where ψ is the left derivative of φ (see (ii) in Remark 1.2). Therefore, for all $t > 1$ we have

$$\ln \frac{\varphi(tx)}{\varphi(x)} = \int_x^{tx} \frac{\psi(y)}{\varphi(y)} dy \geq \int_x^{tx} \frac{\beta}{y} dy = \beta \ln t \quad \forall x > 0.$$

This implies $D(t) \geq t^\beta$. Hence

$$\inf_{t\geq 1} \frac{1 + D(t)}{t} \geq \inf_{t>1} \frac{1 + t^\beta}{t} > 1.$$

Then it follows from

$$\inf_{1>t\geq 1/2} \frac{1 + D(t)}{t} \geq \inf_{1>t\geq 1/2} (1 + D(t)) \geq 1 + D(1/2) > 1$$

and

$$\inf_{1/2\geq t>0}\frac{1+D(t)}{t}\geq 2,$$

that

$$C_1=\inf_{t>0}\frac{1+D(t)}{t}>1.$$

The proof is complete. □

We have the following result:

Lemma 2.8. *Let φ be an Orlicz function with continuous left derivative ψ. Put*

$$H(k)=\sup_{x>0}\frac{\varphi(kx)}{\varphi(x)},\quad a=\sup_{x>0}\frac{x\psi(x)}{\varphi(x)},\quad b=\inf_{x>0}\frac{x\psi(x)}{\varphi(x)}.$$

Then H has the left derivative and the right derivative at 1 and $H'_+(1)=a$, $H'_-(1)=b$.

Proof. For $k>1, x>0$ we have

$$\ln\frac{\varphi(kx)}{\varphi(x)}=\int_x^{kx}\frac{\psi(t)}{\varphi(t)}dt\leq\int_x^{kx}\frac{a}{t}dt=a\ln k.$$

Thus $H(k)\leq k^a$. Hence

$$\limsup_{k\to 1^+}\frac{H(k)-H(1)}{k-1}\leq\lim_{k\to 1^+}\frac{k^a-1}{k-1}=a. \tag{16}$$

Otherwise, let $c\in(0,a)$. There exist $x_0>0,\delta>0$ such that

$$\frac{x\psi(x)}{\varphi(x)}>c\quad\forall x\in(x_0,x_0+\delta).$$

For $k\in(1,1+\frac{\delta}{x_0})$, we have $(x_0,kx_0)\subset(x_0,x_0+\delta)$, and then

$$\ln\frac{\varphi(kx_0)}{\varphi(x_0)}=\int_{x_0}^{kx_0}\frac{\psi(t)}{\varphi(t)}dt\geq\int_{x_0}^{kx_0}\frac{c}{t}dt=c\ln k.$$

This implies that

$$H(k)\geq\frac{\varphi(kx_0)}{\varphi(x_0)}\geq k^c.$$

Hence

$$\liminf_{k\to 1^+}\frac{H(k)-1}{k-1}\geq\lim_{k\to 1^+}\frac{k^c-1}{k-1}=c.$$

Letting $c\to a$ and using (16), we see that H has the right derivative at 1 and $H'_+(1)=a$. Next, we prove that $H'_-(1)=b$. Indeed, for $k<1$ we have

$$\ln\frac{\varphi(x)}{\varphi(kx)}=\int_{kx}^{x}\frac{\psi(t)}{\varphi(t)}dt\geq\int_{kx}^{x}\frac{b}{t}dt=-b\ln k=-\ln k^b\quad\forall x>0,$$

which gives $H(k) \leq k^b$. Hence

$$\liminf_{k\to 1^-} \frac{1-H(k)}{1-k} \geq \lim_{k\to 1^-} \frac{1-k^b}{1-k} = b. \tag{17}$$

On the other hand, for $d > b$, there exists $x_0 > 0$ satisfying

$$\frac{x_0\psi(x_0)}{\varphi(x_0)} < d,$$

and then there exists $\delta > 0$ such that

$$\frac{x\psi(x)}{\varphi(x)} < d \quad \forall x \in (x_0-\delta, x_0).$$

For $1-\frac{\delta}{x_0} < k < 1$ we get $(kx_0, x_0) \subset (x_0-\delta, x_0)$, and then

$$\ln\frac{\varphi(x_0)}{\varphi(kx_0)} = \int_{kx_0}^{x_0} \frac{\psi(t)}{\varphi(t)}dt \leq \int_{kx_0}^{x_0} \frac{d}{t}dt = -\ln k^d.$$

It follows that

$$H(k) \geq \frac{\varphi(kx_0)}{\varphi(x_0)} \geq k^d \quad \forall k \in (1-\frac{\delta}{x_0}, 1).$$

Therefore,

$$\limsup_{k\to 1^-} \frac{1-H(k)}{1-k} \leq \lim_{k\to 1^+} \frac{1-k^d}{1-k} = d \quad \forall d > b. \tag{18}$$

Combining (17) and (18), we obtain that H has the left derivative at 1 and $H'_-(1) = b$. The proof is complete. □

Theorem 2.9. *Let φ be an Orlicz function and its left derivative ψ be continuous. Then $C_2 = 2$ if and only if*

$$\inf_{x>0} \frac{x\psi(x)}{\varphi(x)} \leq 2 \leq \sup_{x>0} \frac{x\psi(x)}{\varphi(x)}. \tag{19}$$

Proof. Necessary. Assume that $C_2 = 2$, we have to show (19) . Indeed, put $g(k) = (1+H(k))/k$. Then $g(1) = 2$ and due to Theorem 2.6, we get $C_2 \leq \inf\{g(k) : k > 0\}$. So, g(1)= $\min\{g(k) : \quad k > 0\}$. Since H has the left derivative and the right derivative at 1, g also has these derivatives at 1. Moreover, it follows from $g(t) \geq g(1) \quad \forall t > 0$ that $g'_+(1) \geq 0 \geq g'_-(1)$. Thus

$$H'_+(1) \geq 2 \geq H'_-(1).$$

From this and using Lemma 2.8, we have (19).
Sufficiency. Assume that (19) is true, we need to prove $C_2 = 2$. Indeed, for all $\epsilon \in (0,1)$, by the continuity of ψ, there exists $x_0 > 0$ such that

$$\frac{x_0\psi(x_0)}{\varphi(x_0)} \in (2-\epsilon, 2+\epsilon).$$

Put

$$f(x) = x_0\chi_{(0,t)}(x), \quad g(x) = \psi(x_0)\omega(x)\chi_{(0,t)}(x),$$

where t is chosen such that $\varphi(x_0)\int_0^t \omega(x)dx = 1-\epsilon$. Hence

$$\int_{\mathbb{R}} \varphi(|f(x)|)\omega(x)dx = 1-\epsilon$$

and

$$\begin{aligned}\int_{\mathbb{R}} f(x)g(x)dx &= \int_0^t x_0\psi(x_0)\omega(x)dx \\ &= \frac{x_0\psi(x_0)}{\varphi(x_0)}\int_0^t \varphi(x_0)\omega(x)dx \in ((2-\epsilon)(1-\epsilon),(2+\epsilon)(1-\epsilon)).\end{aligned}$$

Thus

$$2-3\epsilon < \int_{\mathbb{R}} f(x)g(x)dx < 2-\epsilon.$$

Using Young's equality, we get

$$\int_{\mathbb{R}} f(x)g(x)dx = \int_I \varphi(f^*(x))\omega(x)dx + \int_I \varphi_*(\frac{g^*(x)}{\omega(x)})\omega(x)dx.$$

Then it follows from $\int_I \varphi(|f^*(x)|)\omega(x)dx = 1-\epsilon$ that

$$\int_I \varphi_*(\frac{g^*(x)}{\omega(x)})\omega(x)dx \le 1.$$

So, we obtain

$$\|f\|_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} \le 1, \quad \|g\|_{M^{\mathbb{R}}_{\varphi_*,\omega}} \le 1 \quad \text{and} \quad \int_{\mathbb{R}} f(x)g(x)dx > 2-3\epsilon.$$

Hence

$$C_2 \ge \frac{\|f\|^1_{\Lambda^{\mathbb{R}}_{\varphi,\omega}}}{\|f\|_{\Lambda^{\mathbb{R}}_{\varphi,\omega}}} \ge \int_{\mathbb{R}} f(x)g(x)dx > 2-3\epsilon.$$

Letting $\epsilon \to 0$ we get $C_2 \ge 2$. So, $C_2 = 2$. The proof is complete. □

Theorem 2.10. *Let φ be an Orlicz function. For each $g(x) \in M^{\mathbb{R}}_{\varphi_*,\omega}$, we define*

$$\|g\|^1_{M^{\mathbb{R}}_{\varphi_*,\omega}} = \sup\{\int_{\mathbb{R}} |f(x)g(x)|dx : \quad \|f\|_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} \le 1\}. \tag{20}$$

Then we have the following dual equality

$$\|f\|_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} = \sup\{\int_{\mathbb{R}} |f(x)g(x)|dx : \quad \|g\|^1_{M^{\mathbb{R}}_{\varphi_*,\omega}} \le 1\}. \tag{21}$$

Proof. From (20), we obtain the following inequality

$$\int_{\mathbb{R}} |f(x)g(x)|dx \leq \|f\|_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} \|g\|^1_{M^{\mathbb{R}}_{\varphi_*,\omega}}.$$

Therefore,

$$\|f\|_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} \geq \sup\{\int_{\mathbb{R}} |f(x)g(x)|dx : \quad \|g\|^1_{M^{\mathbb{R}}_{\varphi_*,\omega}} \leq 1\}. \tag{22}$$

Next, we prove the inverse inequality

$$\|f\|_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} \leq \sup\{\int_{\mathbb{R}} |f(x)g(x)|dx : \quad \|g\|^1_{M^{\mathbb{R}}_{\varphi_*,\omega}} \leq 1\}. \tag{23}$$

We can assume that $\|f\|_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} = 1$.
If $f(x)$ is a simple function, then for any $\epsilon > 0$ we have

$$\int_I \varphi((1+\epsilon)f^*(x))\omega(x)dx > 1.$$

Put

$$g(x) = \frac{\psi((1+\epsilon)f^*(x))\omega(x)}{1 + \int_I \varphi_*(\psi((1+\epsilon)f^*(x))\omega(x)dx} \chi_(0,\infty)$$

($g(x)$ is well-defined because $\psi(f^*(x))$ is a simple function too, so we have $\int_I \varphi_*(\psi((1+\epsilon)f^*(x))\omega(x)dx < \infty$). Using Young's inequality, we have

$$\begin{aligned}
\|g\|^1_{M^{\mathbb{R}}_{\varphi_*,\omega}} &= \sup\{\int_{\mathbb{R}} |h(x)g(x)|dx : \quad \|h\|_{\Lambda^{\mathbb{R}}_{\varphi,\omega}} \leq 1\} \\
&\leq \frac{\int_I \psi((1+\epsilon)f^*(x))h^*(x)\omega(x)dx}{1 + \int_I \varphi_*(\psi((1+\epsilon)f^*(x))\omega(x)dx} \\
&\leq \frac{\int_I \psi((1+\epsilon)f^*(x))h^*(x)\omega(x)dx}{\int_I \varphi(h^*(x))\omega(x)dx + \int_I \varphi_*(\psi((1+\epsilon)f^*(x)))\omega(x)dx} \leq 1.
\end{aligned}$$

Thus we get

$$\sup\{\int_{\mathbb{R}} |f(x)h(x)|dx : \quad \|h\|^1_{M^{\mathbb{R}}_{\varphi_*,\omega}} \leq 1\}$$

$$\geq \int_I f^*(x)g^*(x)dx = \frac{\int_I \psi((1+\epsilon)f^*(x))f^*(x)\omega(x)dx}{1+\int_I \varphi_*(\psi((1+\epsilon)f^*(x))\omega(x)dx}$$

$$= \frac{1}{1+\epsilon}\frac{\int_I \varphi((1+\epsilon)f^*(x))\omega(x)dx + \int_I \varphi_*(\psi((1+\epsilon)f^*(x)))\omega(x)dx}{1+\int_I \varphi_*(\psi((1+\epsilon)f^*(x))\omega(x)dx}$$

$$\geq \frac{1}{1+\epsilon} \quad \forall \epsilon > 0.$$

Hence

$$\sup\{\int_{\mathbb{R}} |f(x)h(x)|dx : \quad \|h\|^1_{M^{\mathbb{R}}_{\varphi_*,\omega}} \leq 1\} \geq 1 = \|f\|_{\Lambda^{\mathbb{R}}_{\varphi,\omega}}.$$

Therefore, (23) is true for simple functions $f(x)$.
If $f(x)$ is an arbitrary function, then approximating $f(x)$ by a sequence of simple functions we get (23). Combining (22) with (23), we have (21). The proof is complete. □

3. The Kolmogorov inequality in Orlicz-Lorentz space

The Landau-Kolmogorov inequality

$$\|f^{(k)}\|^n_\infty \leq K(k,n)\|f\|^{n-k}_\infty \|f^{(n)}\|^k_\infty, \tag{24}$$

where $0 < k < n$, is well known and has many interesting applications and generalizations (see [1, 3, 4, 5, 6, 7, 20, 21, 22, 23]). Its study was initiated by Landau [17] and Hadamard [8] (the case $n = 2$). For functions on the whole real line $\mathbb{R}$, Kolmogorov [15] succeeded in finding in explicit form the best possible constants $K(k,n) = C_{k,n}$ in (24), and Stein proved in [22] that inequality (24) still holds for L_p-norm, $1 \leq p < \infty$, with these constants (the same situation also happens for an arbitrary Orlicz norm [1]). In this section will prove that the Kolmogorov inequality still holds for the Orlicz norm and the Luxemburg norm in Orlicz-Lorentz spaces. For simplicity of notations, we we denote $\Lambda^{\mathbb{R}}_{\varphi,\omega}$ by $\Lambda_{\varphi,\omega}$, $\|.\|_{\Lambda^{\mathbb{R}}_{\varphi,\omega}}$ by $\|.\|_{\Lambda_{\varphi,\omega}}$, $\|.\|^1_{\Lambda^{\mathbb{R}}_{\varphi,\omega}}$ by $\|.\|_{\varphi,\omega}$, $M^{\mathbb{R}}_{\varphi_*,\omega}$ by $M_{\varphi_*,\omega}$ and $\|.\|_{M^{\mathbb{R}}_{\varphi_*,\omega}}$ by $\|.\|_{M_{\varphi_*,\omega}}$. Note that if ω is regular (that is $\int_0^t \omega(s)ds \asymp t\omega(t)$), then $M_{\varphi_*,\omega}$ is a linear space, and $\|\cdot\|_{M_{\varphi_*,\omega}}$ is a quasi-norm. Especially, $\Lambda_{\varphi,\omega} \subset S'(\mathbb{R})$ is the space of all tempered generalized functions this follow from the fact that $\int_0^t \omega(s)ds \asymp t\omega(t)$.

We have the following lemmas:

Lemma 3.1. *Let $f \in \Lambda_{\varphi,\omega}$ and $g \in L_1(\mathbb{R})$. Then $f * g \in \Lambda_{\varphi,\omega}$ and*

$$\|f * g\|_{\varphi,\omega} \leq \|f\|_{\varphi,\omega}\|g\|_1$$

and

$$\|f * g\|_{\Lambda_{\varphi,\omega}} \leq \|f\|_{\Lambda_{\varphi,\omega}} \|g\|_1.$$

Lemma 3.2. *Let $n \geq 1$. If $f \in L_{1,loc}(\mathbb{R})$ and its generalized n^{th} derivative $g \in L_{1,loc}(\mathbb{R})$, then f can be redefined on a set of measure zero so that $f^{(n-1)}$ is absolutely continuous and $f^{(n)} = g$ a.e. on $\mathbb{R}$.*

Now, we state our theorem.

Theorem 3.3. *Let φ be an arbitrary Orlicz function, ω be a weight function, f and its generalized derivative $f^{(n)}$ be in $\Lambda_{\varphi,\omega}$. Then $f^{(k)} \in \Lambda_{\varphi,\omega}$ for all $0 < k < n$ and*

$$\|f^{(k)}\|^n_{\varphi,\omega} \leq C_{k,n} \|f\|^{n-k}_{\varphi,\omega} \|f^{(n)}\|^k_{\varphi,\omega}\ , \tag{25}$$

where $C_{k,n}$ are the best constants defined in the Kolmogorov inequality (for the case $p = \infty$).

Proof. We begin to prove (25) with the assumption that $f^{(k)} \in \Lambda_{\varphi,\omega}$, with $k = 0, 1, ..., n$. Indeed, fix $0 < k < n$ and let $\epsilon > 0$ be given. We choose a function $v_\epsilon \in M_{\varphi*,\omega}$, $\|v_\epsilon\|_{M_{\varphi*,\omega}} \leq 1$ such that

$$\Big|\int_{\mathbb{R}} f^{(k)}(x) v_\epsilon(x) dx\Big| \geq \|f^{(k)}\|_{\varphi,\omega} - \epsilon. \tag{26}$$

Put

$$F_\epsilon(x) = \int_{\mathbb{R}} f(x+y) v_\epsilon(y) dy.$$

Then $F_\epsilon(x) \in L_\infty(\mathbb{R})$ by the definition of the Orlicz norm, and

$$F^{(r)}_\epsilon(x) = \int_{\mathbb{R}} f^{(r)}(x+y) v_\epsilon(y) dy, \quad 0 \leq r \leq n \tag{27}$$

in the distribution sense. Actually, for every function $\psi(x) \in C_0^\infty(\mathbb{R})$ it follows from the assumption and the definition of the Orlicz norm that

$$\begin{aligned}
\langle F_\epsilon^{(r)}(x), \psi(x)\rangle &= (-1)^r \langle F_\epsilon(x), \psi^{(r)}(x)\rangle \\
&= (-1)^r \int\limits_{\mathbb{R}} \left(\int\limits_{\mathbb{R}} f(x+y) v_\epsilon(y) dy \right) \psi^{(r)}(x) dx \\
&= (-1)^r \int\limits_{\mathbb{R}} v_\epsilon(y) \left(\int\limits_{\mathbb{R}} f(x+y) \psi^{(r)}(x) dx \right) dy \\
&= \int\limits_{\mathbb{R}} v_\epsilon(y) \left(\int\limits_{\mathbb{R}} f^{(r)}(x+y) \psi(x) dx \right) dy \\
&= \int\limits_{\mathbb{R}} \left(\int\limits_{\mathbb{R}} f^{(r)}(x+y) v_\epsilon(y) dy \right) \psi(x) dx \\
&= \langle \int\limits_{\mathbb{R}} f^{(r)}(x+y) v_\epsilon(y) dy \ , \ \psi(x)\rangle.
\end{aligned}$$

So we have proved (27).
Since $\|v_\epsilon\|_{M_{\varphi*,\omega}} \leq 1$, clearly, for all $x \in \mathbb{R}$,

$$|F_\epsilon^{(r)}(x)| \leq \|f^{(r)}(x+\cdot)\|_{\varphi,\omega} \|v_\epsilon\|_{M_{\varphi*,\omega}} \leq \|f^{(r)}\|_{\varphi,\omega}.$$

Now, we prove the continuity of $F_\epsilon^{(r)}$ on $\mathbb{R}$. Indeed, put $h(x) = f^{(r)}(x), h_t(x) = f^{(r)}(x+t)$, $g(x) = v_\epsilon(x)$. So, to prove the continuity of $F_\epsilon^{(r)}$ on $\mathbb{R}$ we only have to show that

$$\lim_{t\to 0} \int\limits_{\mathbb{R}} (h_t(x) - h(x)) g(x) dx = 0. \tag{28}$$

To do this, it is sufficient to prove for real nonnegative value functions $h(x)$. Since $h \in \Lambda_{\varphi,\omega}$ and $g \in (\Lambda_{\varphi,\omega})'$, we have

$$\int\limits_0^\infty h^*(x) g^*(x) dx < \infty. \tag{29}$$

We first prove (28) whenever $h(x) = \chi_A(x)$ is the characteristic function of the measurable set A, there are two cases, that is
Case 1: $m(A) < +\infty$. We denote $A - t := \{x - t : \quad x \in A\}$, $C_t := A\Delta(A-t)$. Then it follows from $m(A) < \infty$ that $\lim\limits_{t\to 0} m(C_t) = \lim\limits_{t\to 0} m(A\Delta(A-t)) = 0$. We

have $h^*(x) = \chi_{(0,m(A))}(x)$. From this and (29) we get

$$\int_0^{m(A)} g^*(x)dx < +\infty.$$

Hence, for any $\epsilon > 0$, there exists $\delta > 0$ such that $\int_0^{\delta} g^*(x)dx < \epsilon$, and then there is $t_0 > 0$ such that $m(A\Delta(A-t)) < \delta$ for all $|t| < t_0$. Therefore, for $|t| < t_0$:

$$\begin{aligned}\Big|\int_{\mathbb{R}}(\chi_A(x+t)-\chi_A(x))g(x)dx\Big| &\leq \int_{\mathbb{R}}\Big|(\chi_A(x+t)-\chi_A(x))g(x)\Big|dx\\ &= \int_{C_t}\Big|\chi_{C_t}(x)\Big|.\Big|g(x)\Big|dx \leq \int_0^{m(C_t)} g^*(x)dx < \epsilon.\end{aligned}$$

That is

$$\lim_{t\to 0}\int_{\mathbb{R}}(\chi_A(x+t)-\chi_A(x))g(x)dx = 0.$$

Case 2: $m(A) = +\infty$. Then $h^*(x) \equiv 1$ on I. Therefore, from (29), we see that $g^*(x)$ is integrable on I. Thus, $g(x) \in L^1(\mathbb{R})$, and then $\lim_{t\to 0}\|g - g_{-t}\|_{L^1(\mathbb{R})} = 0$. Therefore, it follows from

$$\begin{aligned}\int_{\mathbb{R}}(\chi_A(x+t)-\chi_A(x))g(x)dx &= \int_{\mathbb{R}}\chi_A(x)g(x-t)dx - \int_{\mathbb{R}}\chi_A(x)g(x)dx\\ &= \int_{\mathbb{R}}\chi_A(x)(g(x-t)-g(x))dx\\ &\leq \int_{\mathbb{R}}|(g(x-t)-g(x))|dx = \|g_{-t}-g\|_{L^1(\mathbb{R})}\end{aligned}$$

that

$$\lim_{t\to 0}\int_{\mathbb{R}}(\chi_A(x+t)-\chi_A(x))g(x)dx = 0,$$

i.e., (28) is true for $h(x) = \chi_A(x)$ being the characteristic function of the measurable set A.

By the linearity of integral, (28) is true for all simple functions $h(x)$ satisfying the condition of the theorem.

If $h(x)$ is a nonnegative, measurable function, we consider the sequence of functions $\{h_n(x)\}_{n=1}^{\infty}$ as follows

$$h_n(x) = \sum_{k=0}^{n2^n-1}\frac{k}{2^n}\chi_{A_{n,k}}(x) + n\chi_{A_n}(x),$$

where $A_{n,k} = \{x : \frac{k}{2^n} \le h(x) < \frac{k+1}{2^n}\}$ and $A_n = \{x : h(x) \ge n\}$. Then it is easy to check that $h_n(x) \uparrow h(x)$ a.e., and $\lim_{n\to\infty} m(A_n) = 0$. Given $\epsilon > 0$ and $\delta > 0$. We choose n_0 such that $1/2^n < \epsilon$ and $m(A_n) < \delta$ for all $n \ge n_0$, then $\{x : h(x) - h_n(x) \ge \epsilon\} \subset A_n$. Hence

$$m(\{x : \quad |h(x) - h_n(x)| \ge \epsilon\}) \le m(A_n) < \delta.$$

That is $h_n \xrightarrow{m} f$. So $(h_n - h)^*(x) \to 0$. By Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \int_I (h_n - h)^*(x) g^*(x) dx = 0.$$

Then it follows from

$$\begin{aligned}\Big|\int_{\mathbb{R}} (h(x+t) - h(x)) g(x) dx\Big| &= \Big|\int_{\mathbb{R}} (h(x+t) - h_n(x+t)) g(x) dx \\ &+ \int_{\mathbb{R}} (h_n(x+t) - h_n(x)) g(x) dx + \int_{\mathbb{R}} (h_n(x) - h(x)) g(x) dx\Big| \\ &\le 2 \int_I (h_n - h)^*(x) g^*(x) dx + \Big|\int_{\mathbb{R}} (h_n(x+t) - h_n(x)) g(x) dx\Big|\end{aligned}$$

that

$$\limsup_{t\to 0} \Big|\int_{\mathbb{R}} (h(x+t) - h(x)) g(x) dx\Big| \le 2 \int_0^\infty (h_n - h)^*(x) g^*(x) dx \quad \forall n \in \mathbb{N}.$$

Hence

$$\limsup_{t\to 0} \Big|\int_{\mathbb{R}} (h(x+t) - h(x)) g(x) dx\Big| \le 2 \lim_{n\to\infty} \int_0^\infty (h_n - h)^*(x) g^*(x) dx = 0.$$

This gives

$$\lim_{t\to 0} \int_{\mathbb{R}} (h(x+t) - h(x)) g(x) dx = 0.$$

So, (28) has been proved.

The functions $F_\epsilon^{(r)}$ are continuous and bounded on $\mathbb{R}$, therefore it follows from the Landau-Kolmogorov inequality and (26)-(27) that

$$\big(\|f^{(k)}\|_{\varphi,\omega} - \epsilon\big)^n \le |F_\epsilon^{(k)}(0)|^n \le \|F_\epsilon^{(k)}\|_\infty^n \le C_{k,n} \|F_\epsilon\|_\infty^{n-k} \|F_\epsilon^{(n)}\|_\infty^k.$$

On the other hand,

$$\begin{aligned}\|F_\epsilon\|_\infty &\le \|f(x+\cdot)\|_{\varphi,\omega} \|v_\epsilon(\cdot)\|_{M_{\varphi_*,\omega}} \le \|f\|_{\varphi,\omega}\ , \\ \|F_\epsilon^{(n)}\|_\infty &\le \|f^{(n)}(x+\cdot)\|_{\varphi,\omega} \|v_\epsilon(\cdot)\|_{M_{\varphi_*,\omega}} \le \|f^{(n)}\|_{\varphi,\omega}\ .\end{aligned}$$

Hence

$$\big(\|f^{(k)}\|_{\varphi,\omega} - \epsilon\big)^n \le C_{k,n} \|f\|_{\varphi,\omega}^{n-k} \|f^{(n)}\|_{\varphi,\omega}^k.$$

By letting $\epsilon \to 0$, we have (25).

To complete the proof, it remains to show that $f^{(k)} \in \Lambda_{\varphi,\omega}$, with $1 \le k \le n-1$ if $f, f^{(n)} \in \Lambda_{\varphi,\omega}$. Indeed, by Lemma 3.2 we can assume that $f, f', \dots, f^{(n-1)}$ are continuous on $\mathbb{R}$ and $f^{(n-1)}$ is absolutely continuous on $\mathbb{R}$, because $f, f^{(n)} \in \Lambda_{\varphi,\omega}$. Let $\psi \in C_0^\infty(\mathbb{R}), \psi \ge 0, \psi(x) = 0$ for $|x| \ge 1$ and $\int\limits_{\mathbb{R}} \psi(x)dx = 1$. We put $\psi_\lambda(x) = 1/\lambda\psi(x/\lambda), \lambda > 0$ and $f_\lambda = f * \psi_\lambda$. Then $f_\lambda \in C^\infty(\mathbb{R})$ and $f_\lambda^{(k)} = f * \psi_\lambda^{(k)} = f^{(k)} * \psi_\lambda, k \ge 0$. It follows from Lemma 3.1 that $f_\lambda^{(k)} \in \Lambda_{\varphi,\omega}$. Then by the fact proved above, we obtain

$$\|f_\lambda^{(k)}\|_{\varphi,\omega}^n \le C_{k,n}\|f_\lambda\|_{\varphi,\omega}^{n-k}\|f_\lambda^{(n)}\|_{\varphi,\omega}^k, \quad 0 < k < n.$$

It follows from the following inequalities

$$\|f_\lambda\|_{\varphi,\omega} \le \|f\|_{\varphi,\omega}\|\psi_\lambda\|_1 = \|f\|_{\varphi,\omega},$$

$$\|f_\lambda^{(n)}\|_{\varphi,\omega} \le \|f^{(n)}\|_{\varphi,\omega}\|\psi_\lambda\|_1 = \|f^{(n)}\|_{\varphi,\omega}$$

that the set $\{f_\lambda^{(k)}\}_{\lambda\in\mathbb{R}_+}$ is bounded in $\Lambda_{\varphi,\omega}$ and, by the continuity of $f_\lambda^{(k)}$, $\lim\limits_{\lambda\to 0} f_\lambda^{(k)}(x) = \lim\limits_{\lambda\to 0} f^{(k)} * \psi_\lambda = f^{(k)}(x) \forall x \in \mathbb{R}$. Indeed, for all $x \in \mathbb{R}$, we have

$$\begin{aligned} |f_\lambda^{(k)}(x) - f^{(k)}(x)| &= \Big|\int\limits_{\mathbb{R}} \big(f^{(k)}(x-y) - f^{(k)}(y)\big)\psi_\lambda(y)dy\Big| \\ &\le \int\limits_{|\lambda|\le\epsilon} \big|f^{(k)}(x-y) - f^{(k)}(y)\big|\psi_\lambda(y)dy \\ &\le \sup_{|y|\le\lambda} \big|f^{(k)}(x-y) - f^{(k)}(y)\big| \to 0 \quad as \quad \lambda \to 0^+. \end{aligned}$$

Put $g_\lambda(x) = \inf\limits_{0<\mu\le\lambda} |f_\mu^{(k)}(x)|$. Then $g_\lambda \in \Lambda_{\varphi,\omega}$ for all $\lambda \in \mathbb{R}_+$, the set $\{g_\lambda\}_{\lambda\in\mathbb{R}_+}$ is bounded in $\Lambda_{\varphi,\omega}$ and $g_\lambda \uparrow |f^{(k)}|$ as $\lambda \to 0^+$. Therefore, $g_\lambda^* \uparrow {f^{(k)}}^*$ as $\lambda \to 0^+$. Choose $M > 0$ such that $\|g_\lambda\| < M$ for all $\lambda \in \mathbb{R}_+$. So,

$$\int\limits_0^\infty \varphi(\frac{g_\lambda^*(t)}{M})\omega(t)dt \le 1 \quad \forall\lambda \in \mathbb{R}_+.$$

Letting $\lambda \to 0^+$, by the monotone convergence theorem, we get

$$\int\limits_0^\infty \varphi(\frac{{f^{(k)}}^*(t)}{M})\omega(t)dt \le 1.$$

Hence $f^{(k)} \in \Lambda_{\varphi,\omega}$ for $1 \le k \le n-1$. The proof is complete. $\square$

From the proof of (28) we have the following result.

Proposition 3.4. *Let f, g be measurable functions satisfying the following condition*

$$\int_0^\infty f^*(x)g^*(x)dx < +\infty.$$

Then

$$\lim_{t\to 0}\int_{\mathbb{R}} (f(x+t) - f(x))g(x)dx = 0.$$

From Proposition 3.4 , we have the following:

Corollary 3.5. *Let $f \in \Lambda^{\mathbb{R}}_{\varphi,\omega}, g \in M^{\mathbb{R}}_{\varphi*,\omega}$. Then*

$$\lim_{t\to 0}\int_{\mathbb{R}} (f(x+t) - f(x))g(x)dx = 0.$$

For the Luxemburg norm $\|\cdot\|_{\Lambda_{\varphi,\omega}}$, the Kolmogorov inequality also holds:

Theorem 3.6. *Let φ be an arbitrary Orlicz function, ω be a weight function, f and its generalized derivative $f^{(n)}$ be in $\Lambda_{\varphi,\omega}$. Then $f^{(k)} \in \Lambda_{\varphi,\omega}$ for all $0 < k < n$ and*

$$\|f^{(k)}\|^n_{\Lambda_{\varphi,\omega}} \le C_{k,n}\|f\|^{n-k}_{\Lambda_{\varphi,\omega}}\|f^{(n)}\|^k_{\Lambda_{\varphi,\omega}}.$$

Reference

[1] H. H. Bang, A remark on the Kolmogorov-Stein inequality, *J. Math. Anal. Appl.* **203** (1996), 861-867.

[2] H. H. Bang, N. V. Hoang and V.N. Huy, Best constants for the inequalities between equivalent norms in Orlicz spaces, *Bull. Polish Akad. Sci.* (2011) (in the Press).

[3] B. Bollobas, The spatial numerical range and powers of an operator, *J. London. Math. Soc.* **7** (1973), 435-440.

[4] M. W. Certain and T. G. Kurtz, Landau-Kolmogorov inequalities for semigroups and groups, *Proc. Amer. Math. Soc.* **63** (2) (1977), 226-230.

[5] P. R. Chernov, Optimal Landau-Kolmogorov inequalities for dissipative operators in Hilbert and Banach spaces, *Adv. in Math.* **4** (3) (1979), 137-144.

[6] R. A. DeVore and G. G. Lorentz, *Constructive Approximation*, Springer-Verlag, Berlin, 1993.

[7] Z. Ditzian, Some remarks on inequalities of Landau and Kolmogorov, *Aequationes Math.* **12** (1975), 145-151.

[8] J. Hadamard, Sur le module maximum d'une fonction et de ses dérivées, C. 17. *Soc. Math. France* **41** (1914), 68-72.

[9] H. Hudzik and A. Kaminska, Monotonicity properties of Lorentz spaces, *Proc. Amer. Math. Soc.* **123**(9) (1995), 2715-2721.

[10] H. Hudzik, A. Kaminska and M. Mastylo,Geometric properties of some Calderon-Lozanovskii spaces and Orlicz- Lorentz spaces, *Houston J. Math.* **22** (1996), 639- 663.

[11] H. Hudzik, A. Kaminska and M. Mastylo, On the dual of Orlicz-Lorentz space, *Proc. Amer. Math. Soc.* **130**(6) (2002), 1645-1654.

[12] A. Kaminska, Some remarks on Orlicz-Lorentz spaces, *Math. Nachr.* **147** (1990), 29-38.

[13] A. Kaminska and L. Maligranda, On Lorentz spaces $\Gamma_{p,\omega}$, *Israel J. Math.* **140** (2004), 285-318.

[14] A. Kaminska, Some remarks on Orlicz- Lorentz spaces, *Math. Nachr.* **147** (1990), 29-38.

[15] A. N. Kolmogorov, On inequalities between upper bounds of the successive derivatives of an arbitrary function on an infinite interval, *Amer. Math. Soc. Transl.* **2** (1) (1962), 233-243.

[16] S. G. Krein, Ju. I. Petunia and E. M. Semenov, *Interpolation of Linear Operators*, Amer. Math. Soc. Trans. of Math. Monog. 54, Providence, RI, 1982.

[17] E. Landau, Einige Ungleichungen für zweimal differenzierbare Funktionen, *Proc. London Math. Soc.* **13** (1913), 43-49.

[18] P. K. Lin and H. Sun, Some geometrical properties of Orlicz-Lorentz spaces, *Arch. Math.* **64** (1995), 500-511.

[19] M.M. Rao and Z.D. Ren, *Theory of Orlicz spaces*, Marcel Dekker Inc., New York, 1991.

[20] S. B. Stechkin, Best approximation of linear operators, *Mat. Zametki* **1** (1967), 137-148.

[21] S. B. Stechkin, On the inequalities between the upper bounds of the derivatives of an arbitrary function on the hairline, *Mat. Zametki* **1** (1967), 665-674.

[22] E. M. Stein, Functions of exponential type, *Ann. Math.* **65** (1957), 582-592.

[23] V. M. Tikhomirov, G. G. Magaril-lljaev, Inequalities for derivatives, *in Kolmogorov A. N. Selected Papers*, Nauka, Moscow, 1985, 387-390.

Institute of Mathematics,
18 Hoang Quoc Viet Street, Cau Giay, Hanoi, Vietnam
E-mail address: hhbang@math.ac.vn

Institute of Mathematics,
18 Hoang Quoc Viet Street, Cau Giay, Hanoi, Vietnam
E-mail address: vanhoang0610@yahoo.com

Department of Mathematics,
College of Science, Vietnam National University,
334 Nguyen Trai Street, Thanh Xuan, Hanoi, Vietnam
E-mail address: nhat_huy85@yahoo.com