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REMARKS ON NONCLASSICAL SHOCK WAVES FOR VAN DER WAALS FLUIDS

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Dedicated to Professor Tran Duc Van on the occasion of his sixtieth birthday

Abstract. We consider the Riemann problem for isentropic van der Waals fluids, where LeFloch's concept of nonclassical shock waves is studied. Motivated by [3, 32], we study all types of nonclassical shocks including shocks that correspond to the traveling waves of an autonomous system of differential equations with four equilibria resulted from a diffusive-dispersive model. Moreover, the range of kinetic relation is extended to the whole admissible nonclassical shock set. Corresponding to each of the two inflection points of the pressure function we can define a kinetic function. The kinetic functions may not be monotone. It is very interesting that there could be nonclassical shocks that satisfy both a kinetic relation and the Lax shock inequalities. It turns out that nonclassical Riemann solutions may form a two-parameter family of solutions. This raises an open question for the study on the selection of a unique nonclassical solution.

1. Introduction

In this paper we consider the nonclassical shock waves for the following isentropic van der Waals fluid

$$\begin{aligned} &\partial_t v - \partial_x u = 0,\\ &\partial_t u + \partial_x p(v) = 0, \quad x \in \mathbb{R}, t > 0, \end{aligned} \tag{1.1}$$

where $u, v > 0$ and p denote the velocity, specific volume, and the pressure, respectively. The system (1.1) is conservation laws of mass and momentum of isentropic gas dynamics equations in Lagrange coordinates. The system can be obtained from the common gas dynamics equations in Lagrange coordinates by writing the equation of state of the form $p = p(v, S)$, where S is the entropy and assumed to be constant.

In the sequel, the pressure function - as a function of the specific volume - is assumed to be of van der Waals type, of which it changes the concavity twice.

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Precisely, we assume that there are two constants $0 < a < b$ such that

$$
\begin{aligned}
&p''(v) > 0 \quad \text{for } v \in (0,a) \cup (b,+\infty),\\
&p''(v) < 0 \quad \text{for } v \in (a,b),\\
&p'(a) < 0,\\
&\lim_{v\to 0} p(v) = +\infty, \quad \lim_{v\to +\infty} p(v) \geq 0.
\end{aligned}
\tag{1.2}
$$

Under the assumptions (1.2), the pressure is decreasing and admits exactly two inflection points at $v = a$ and $v = b$. As seen later, the system (1.1) is therefore strictly hyperbolic, and the characteristic fields of (1.1) are not genuinely nonlinear.

The Riemann problem for (1.1) is the Cauchy problem with the initial data of the form

$$
(v,u)(x,0) = \begin{cases} (v_L,u_L), & \text{if} \quad x < 0,\\ (v_R,u_R), & \text{if} \quad x > 0, \end{cases}
\tag{1.3}
$$

where $(v_L,u_L),(v_R,u_R)$ are given constant states. Solutions of the Riemann problem consist of a finite number of waves. Therefore, solving the Riemann problem enables us to see the structure of waves. Moreover, Riemann solutions are used in several principle numerical schemes such as the Glimm scheme and the Godunov scheme for the initial-value problem.

A classical shock is a discontinuity of (1.1) that satisfies Liu's entropy condition, (see Liu [25]). The reader is referred to the pioneering work of Lax [15] for the theory of shock waves, developed by Liu [25] for systems having non-genuinely nonlinear characteristic fields. See also recent works [8, 9, 12, 13, 23, 14, 24, 31] for the developments of the shock wave theory to non-conservative systems of balance laws. A nonclassical shock is a discontinuity of (1.1) that satisfies a single entropy inequality and a kinetic relation and violates Liu's entropy condition. The kinetic relation is a relation between the left-hand and the right-hand states of the shock and is characterized by a kinetic function. The existence of nonclassical shock waves agrees with the existence of the corresponding traveling waves, see [5, 6, 4, 7]. Pioneering works on nonclassical shock waves have been done by Abayaratne-Knowles [1, 2], Hayes-LeFloch [10, 11], and LeFloch [16]. See also [22, 29, 28, 26, 27, 35, 18, 17, 33, 34, 30] and the references therein for related works.

In our earlier work LeFloch-Thanh [19], the Riemann problem (1.1)-(1.3) was addressed, where a single kinetic function defining nonclassical shock waves is allowed. This kinetic function is associated with the smaller inflection point $v = a$ of the pressure function, defined up to the larger inflection point $v = b$, and assumed to be decreasing. In particular, a nonclassical shock defined by this kinetic function "crosses" the graph of the pressure exactly once between the left-hand and the right-hand states.

Questions on the kinetic function arise. The first question is on the range of kinetic relations, as whether we can extend the definition of a kinetic function to the whole admissible nonclassical wave set, and whether a kinetic function associated with the larger inflection point of the pressure can be used. More interestingly, the second question is on the monotony of the kinetic function. If we look at the other models in [20, 21], then the boundary functions φ^t, φ_0, which surround the kinetic function, are decreasing. Naturally, the kinetic function in these models are decreasing as well. However, the boundary functions for the model (1.1) are not entirely decreasing. Precisely, the boundary functions for the nonclassical shock set corresponding to the smaller inflection point is first decreasing and then increasing. Meanwhile the boundary functions for the nonclassical shock set corresponding to the larger inflection point is first increasing and then decreasing. Furthermore, the boundary function that corresponds to the vanishing entropy dissipation can be served as a kinetic function. We therefore expect that in general the monotony of a kinetic function also looks like its boundary functions. The third question is whether we can define the kinetic relation in such a way that it can cover nonclassical shocks crossing the graph of the pressure function four times. If so, these nonclassical shocks evidently satisfy both a kinetic relation and the Lax shock inequalities (see Lax [15]). This is motivated by the recent works of Bedjaoui-Chalons-Coquel-LeFloch [3] and our work Thanh [32], where a traveling wave of the diffusive-dispersive model of (1.1) may result an autonomous system of differential equations having exactly four equilibria. Our aim in this paper is to seek for an answer to these above three questions. Moreover, the paper also raises an open question on the uniqueness of nonclassical solutions.

The paper is organized as follows. Section 2 is devoted to basic concepts and results concerning classical and nonclassical shock waves for the isentropic model of a van der Waals fluid. Section 3 is to revisit the description of the sets of classical shocks and nonclassical shocks. Here, we improve the analysis presented earlier in [19]. In Section 4, by allowing both types of kinetic functions to operate and letting kinetic relations to be global, we construct nonclassical solutions of the Riemann problem (1.1)-(1.3). It turns out that two-parameter sets of Riemann solutions can be obtained.

2. Preliminaries

The Jacobian matrix of the system (1.1) is given by

$$A = \begin{pmatrix} p'(v) & 0 \\ 0 & -1 \end{pmatrix}.$$

Under the assumptions (1.2), $p' < 0$. Therefore, the Jacobian matrix of (1.1) admits two distinct eigenvalues

$$\lambda_1(v) := -\sqrt{-p'(v)} < \sqrt{-p'(v)} := \lambda_2(v). \tag{2.1}$$

The corresponding right-eigenvectors can be chosen as

$$r_1(v) = (1, \sqrt{-p'(v)})^T, \quad r_2(v) = (1, -\sqrt{-p'(v)})^T.$$

The system (1.1) is thus strictly hyperbolic. We have

$$\nabla\lambda_1(v) = (\frac{p"(v)}{2\sqrt{-p'(v)}}, 0)^T, \quad \nabla\lambda_2(v) = (-\frac{p"(v)}{2\sqrt{-p'(v)}}, 0)^T.$$

Thus,

$$\begin{aligned}\nabla\lambda_1(v)^T r_1(v) &= \frac{p"(v)}{2\sqrt{-p'(v)}},\\ \nabla\lambda_2(v)^T r_2(v) &= \frac{-p"(v)}{2\sqrt{-p'(v)}},\end{aligned}$$

which vanish exactly twice at $v = a$ and $v = b$. The two characteristic fields thus fail to be genuinely nonlinear along the lines $v = a$ and $v = b$ in the phase domain.

A discontinuity of (1.1) connecting the left-hand state (v_-, u_-) to the right-hand state (v_+, u_+) propagating with speed s is a weak solution of the form

$$(v,u)(x,t) = \begin{cases} (v_-, u_-), & \text{if} \quad x < st,\\ (v_+, u_+), & \text{if} \quad x > st.\end{cases} \tag{2.2}$$

The Rankine-Hugoniot relations for the discontinuity (2.2) read

$$\begin{aligned} -s(v_+ - v_-) - (u_+ - u_-) &= 0,\\ -s(u_+ - u_-) + p(v_+) - p(v_-) &= 0.\end{aligned} \tag{2.3}$$

From (2.3), we can see that the shock speed

$$s = \mp\sqrt{-\frac{p(v_+) - p(v_-)}{v_+ - v_-}}$$

is well-defined and independent of u_- and u, so we simply set $s = s(v_-, v_+)$, where the 1– and 2–shocks correspond to the minus and the plus sign, respectively. The Hugoniot set is thus composed of two Hugoniot curves $\mathcal{H}_1, \mathcal{H}_2$ corresponding to $s < 0$ and $s > 0$:

$$\mathcal{H}_{1,2}: \quad u_+ = u_- \pm \sqrt{-(p(v_+) - p(v_-))(v_+ - v_-)}. \tag{2.4}$$

Let us recall standard entropy criteria for hyperbolic systems of conservation laws. The Lax shock inequalities require that any discontinuity connecting the left-hand state (v_-, u_-) and the right-hand state (v_+, u_+) satisfies

$$\lambda_i(v_+) < s_i(v_+, v_-) < \lambda_i(v_-), \quad i = 1, 2, \tag{2.5}$$

where $s_i = s_i(v_+, v_-)$ stands for the i-shock speed, $i = 1, 2$.

Since both the characteristic fields of the system (1.1) are non-genuinely nonlinear characteristic fields, Lax shock inequalities can be replaced by Liu's entropy condition to ensure the uniqueness. Liu's entropy condition is the one that imposes along Hugoniot curves:

$$s(v_-, v) \geq s(v_+, v_-) \quad \text{for any } v \text{ between } v_+ \text{ and } v_-, \tag{2.6}$$

where $s(v_+, v)$ denotes the speed of the discontinuity connecting v and v_+. Thus, Liu's entropy condition means that any discontinuity connecting the left-hand state (v_-, u_-) and the right-hand state (v_+, u_+) fulfils:

-For 1-shocks it holds that

$$\frac{p(v) - p(v_-)}{v - v_-} \geq \frac{p(v_+) - p(v_-)}{v_+ - v_-}, \qquad \text{for any } v \text{ between } v_+ \text{ and } v_-.$$

Geometrically, the graph of the pressure function is lying below (above) the line segment between $(v_\pm, p(v_\pm))$ if $v_- > v_+$ ($v_- < v_+$, respectively).

-For 2-shocks it holds that

$$\frac{p(v) - p(v_-)}{v - v_-} \leq \frac{p(v_+) - p(v_-)}{v_+ - v_-}, \qquad \text{for any } v \text{ between } v_+ \text{ and } v_-.$$

Geometrically, the graph of the pressure function is lying above (below) the line segment between $(v_\pm, p(v_\pm))$ if $v_- > v_+$ ($v_- < v_+$, respectively).

Observe that Liu's strict entropy condition (where the inequalities "$\geq$" and "$\leq$" above are replaced by the strict inequalities " $>$ " and "$<$", respectively) implies the Lax shock inequalities (2.5).

Definition 2.1. (a) A discontinuity of the form (2.2) is said to be an *admissible shock* if it satisfies the entropy inequality

$$\begin{aligned} &\partial_t U(u, v) + \partial_x F(u, v) \leq 0, \\ &U(u, v) := \frac{u^2}{2} + \Sigma(v), \quad F(u, v) := u\, p(v), \Sigma(v) := -\int_0^v p(w)\, dw, \end{aligned} \tag{2.7}$$

(in the sense of distribution).

(b) An admissible shock satisfying the Liu's entropy condition (2.6) is called a *classical shock*. Otherwise, it is called a *nonclassical shock.*

It is clear that the entropy inequality (2.7) for the discontinuity (2.2) becomes

$$-s(U(u_+, v_+) - U(u_-, v_-)) + (F(u_+, v_+) - F(u_-, v_-)) \leq 0. \tag{2.8}$$

Substituting U, F from (2.7) and simplifying, the inequality (2.8) is equivalent to the condition that the following so-called *entropy dissipation* is non-positive:

$$E(v_-, v_+) := -s(v_-, v_+) \Big(\Sigma(v_+) - \Sigma(v_-) + \frac{p(v_+) + p(v_-)}{2} (v_+ - v_-) \Big) \leq 0. \tag{2.9}$$

Since $s < 0$ for 1-shocks and $s > 0$ for 2-shocks, the entropy inequality (2.7) is thus equivalent to the following entropy condition:

$$\begin{aligned} &D(v_-, v_+) := \int_{v_+}^{v_-} p(w) dw + \frac{p(v_+) + p(v_-)}{2} (v_+ - v_-), \\ &D(v_-, v_+) \leq 0, \quad \text{for 1-shocks}, \\ &D(v_-, v_+) \geq 0, \quad \text{for 2-shocks}. \end{aligned} \tag{2.10}$$

Investigating the entropy condition (2.10) requires several notations and concepts which will be addressed immediately.

In the following we consider points on this graph and refer to them simply by their v-coordinate. In the interval (a, b), the function p is concave, and thus remains above its tangent at the inflection point b. This tangent intersects the graph of p at some other point, outside the interval (a, b), whose coordinate will be denoted by $b^{-t} < a$. Similarly the tangent to the curve at the other inflection point a also intersects the graph of p at some point $a^{-t} > b$.

From any $v \in (b^{-t}, a^{-t})$ we can draw two tangent lines to the graph of the pressure with the corresponding tangency points denoted by $\varphi^t(v) < \psi^t(v)$. In other words, we have

$$p'\big(\varphi^t(v)\big) = \frac{p(v) - p\big(\varphi^t(v)\big)}{v - \varphi^t(v)},$$
$$p'\big(\psi^t(v)\big) = \frac{p(v) - p\big(\psi^t(v)\big)}{v - \psi^t(v)}.$$

See Figure 1.

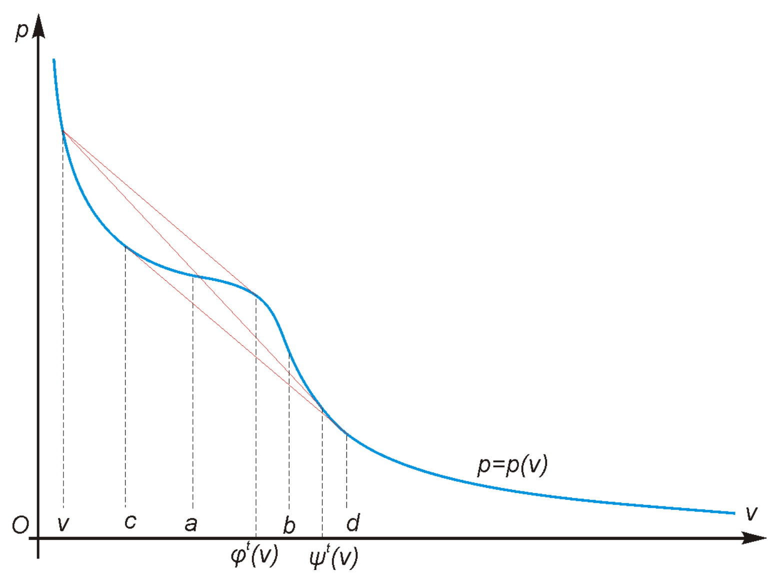

FIGURE 1. The tangent functions φ^t and ψ^t

The definition extends to the end points of the interval under consideration by setting

$$\varphi^t(b^{-t}) = \psi^t(b^{-t}) = b \quad \text{and} \quad \varphi^t(a^{-t}) = \psi^t(a^{-t}) = a.$$

It is easy to see that the values v and $\psi^t(v)$ always lie on different sides with respect to b, and the values v and $\varphi^t(v)$ always lie on different sides with respect to a, in the sense that:

$$(\varphi^t(v) - a)(v - a) < 0 \quad \text{for } v \neq a, \qquad \varphi^t(a) = a,$$
$$(\psi^t(v) - b)(v - b) < 0 \quad \text{for } v \neq b, \qquad \psi^t(b) = b.$$

Moreover, considering the convex hull of the epigraph of p, we see that there exist two points c and d such that

$$b^{-t} < c < a < b < d < a^{-t}$$

and

$$\psi^t(c) = d \quad \text{and} \quad \varphi^t(d) = c.$$

The function ψ^t is increasing for $v \in [b^{-t}, c]$ and decreasing for $v \in [c, a^{-t}]$. The function φ^t is decreasing for $v \in [b^{-t}, d]$ and increasing for $v \in [d, a^{-t}]$. Moreover, φ^t maps $[b^{-t}, a^{-t}]$ onto $[c, b]$, while ψ^t maps $[b^{-t}, a^{-t}]$ onto $[a, d]$.

The points c and d can be characterized by the conditions $c < a < b < d$ and

$$p'(c) = \frac{p(d) - p(c)}{d - c} = p'(d).$$

We observe that the tangent at any point $v \notin [c, d]$ remains globally below the graph of p. So we focus on values $v \in [c, d]$.

For all $v \in (c, d)$, the tangent at the point with coordinate v intersects the graph of p at exactly two distinct points, say denoted by $\varphi^{-t}(v)$ and $\psi^{-t}(v)$ with the convention

$$\varphi^{-t}(v) < \psi^{-t}(v).$$

3. Classification of admissible shock waves

In this section we will review the determination of admissible shocks, classical shocks as well as nonclassical shocks. These results were presented in [19]. However, we present these results here with a simpler way by relying on the simpler entropy condition (2.10) rather than the more complicated (2.9) as in [19]. Observe that our construction is from left-to-right for 1-waves and right-to-left for 2-waves, the entropy inequality (2.10) in both cases of 1-shocks and 2-shocks will be basically the same. Without loss of generality, we consider in the sequel 1-shocks only.

Given a left-hand state (v_-, u_-), we need to determine the set of all right-hand states (v_+, u_+) that can be connected to (v_-, u_-) by an admissible 1-shock, which satisfies the entropy inequality (2.10). We therefore fix an arbitrary v_- and investigate $D(v_-, v_+)$ as a function of v_+. For simplicity we still refer to $D(v_-, v_+)$ as the entropy dissipation if there is no confusion:

$$D(v_-, v_+) = \int_{v_+}^{v_-} p(w)dw + \frac{p(v_+) + p(v_-)}{2}(v_+ - v_-).$$

We have

$$D_{v_+}(v_-, v_+) = (p(v_-) - p(v_+) - p'(v_+)(v_- - v_+))/2.$$

Then, it holds that

$$\begin{aligned} &D_{v_+}(v_-, \varphi^t(v_-)) = D_{v_+}(v_-, \psi^t(v_-)) = 0, \\ &D_{v_+}(v_-, v_+) < 0, \quad \varphi^t(v_-) < v_+ < \psi^t(v_-), \\ &D_{v_+}(v_-, v_+) > 0, \quad v_+ < \varphi^t(v_-), \quad v > \psi^t(v_-)). \end{aligned} \tag{3.1}$$

The entropy dissipation $D(v_-, v_+)$ thus attains a local maximum at $v_+ = \varphi^t(v_-)$ and a local minimum at $v_+ = \psi^t(v_-)$. To determine the sign of the entropy dissipation, we need to know the sign of the local maximum

$$P(v_-) := D(v_-, \varphi^t(v_-)), \tag{3.2}$$

and the sign of the local minimum

$$Q(v_-) := D(v_-, \psi^t(v_-)). \tag{3.3}$$

Properties of the quantity $P(v_-)$ defined by (3.2) are given in the following lemma.

Lemma 3.1. *The function $P = P(v)$ defined by (3.2) for $v \in (b^{-t}, a^{-t})$ is increasing in the interval (a, d) and decreasing in the intervals $(0, a)$ and $(d, +\infty)$. Moreover, there exists exactly one value $f \in (d, a^{-t})$ such that*

$$\begin{aligned} &P(a) = P(f) = 0, \\ &P(v) < 0 \quad \textit{if } v > f, \\ &P(v) < 0, \quad \textit{if } v < f, v \neq a. \end{aligned} \tag{3.4}$$

Proof. The derivative of P is given by

$$\begin{aligned} P'(v) &= \frac{d}{dv} D(v, \varphi^t(v)) = D_{v_-}(v, \varphi^t(v)) + D_{v_+}(v, \varphi^t(v)) \frac{d}{dv} \varphi^t(v) \\ &= D_{v_-}(v, \varphi^t(v)) = \frac{-1}{2} (p(\varphi^t(v)) - p(v) - p'(v)(\varphi^t(v) - v)) \\ &= \frac{(v - \varphi^t(v))}{2} \Big(\frac{p(\varphi^t(v)) - p(v)}{\varphi^t(v) - v} - p'(v) \Big), \end{aligned} \tag{3.5}$$

where the second equation is implied from the first line in (3.1) and the last expression can be understood at $v = a$ as the limit when $v \to a$ and is equal to zero. The following observations can be easily checked.

- If $v < a$, then the first factor of the last expression in (3.5) is negative, and the second factor is positive. Therefore, $P'(v) < 0$.
- If $a < v < d$ then both factors in (3.5) are positive. Thus, $P'(v) > 0$.
- If $v > d$, then the first factor in (3.5) is positive, and the second factor is negative. Therefore, $P'(v) < 0$.

These give the monotony property of $P(v), v > 0$. Clearly, $P(a) = 0$. Since P is decreasing for $v < a$, $P(b^{-t}) > 0$. Let us show that $P(a^{-t}) < 0$. Indeed,

$$P(a^{-t}) = \int_a^{a^{-t}} p(y)dy - \frac{p(a) + p(a^{-t})}{2}(a^{-t} - a).$$

Since the tangent line to the graph of the pressure at $(a, p(a))$ lies above the graph of the pressure in the interval (a, a^{-t}), the area under this tangent line, between a and a^{-t} and above the v-axis given by $\frac{p(a)+p(a^{-t})}{2}(a^{-t} - a)$ is larger than the area under the curve $p = p(v)$, which is given by $\int_a^{a^{-t}} p(y)dy$. This shows that $P(a^{-t}) < 0$. Thus, there is exactly one value $f \in (d, a^{-t})$ such that

$P(f) = 0$. The other conclusions in (3.4) follow immediately. Lemma 3.1 is completely proved. □

Properties of the function Q in (3.3) are given by the following lemma, whose proof is omitted, since it is similar to that of Lemma 3.1.

Lemma 3.2. *The function $Q = Q(v)$ defined by (3.3) for $v \in (b^{-t}, a^{-t})$ is increasing in the interval (c, b) and decreasing in the intervals $(0, c)$ and $(b, +\infty)$. Moreover, there exists exactly one value $e \in (d, a^{-t})$ such that*

$$\begin{aligned} &Q(b) = Q(e) = 0, \\ &Q(v) > 0 \quad \text{if } v < e, \\ &Q(v) < 0, \quad \text{if } v > e, v \neq b. \end{aligned} \tag{3.6}$$

Using Lemmas 3.1 and 3.2, we can establish the following important theorem, which characterizes the set of admissible shock waves.

Theorem 3.3. (Theorem 2.2 [19]) *Given an arbitrary fixed v_-. If $v_- \in (0, b^{-t}) \cup (a^{-t}, +\infty)$, the entropy dissipation $D(v_-, v_+)$ is a decreasing function of $v_+ > 0$. If $v_- \in [b^{-t}, a^{-t}]$, the function $v_+ \mapsto D(v_-, v_+)$ is increasing in the intervals $\big(0, \varphi^t(v_-)\big]$ and $\big[\psi^t(v_-), +\infty\big)$, decreasing in the interval $\big[\varphi^t(v_-), \psi^t(v_-)\big]$, and has a local maximum $P(v_-)$ at $\varphi^t(v_-)$ and a local minimum $Q(v_-)$ at $\psi^t(v_-)$. The functions $P(v), Q(v), v > 0$ are characterized by Lemmas 3.1 and 3.2.*

The following conclusions on admissible shock waves hold.

(a) *If $v_- \in (0, e] \cup [f, +\infty)$, then (2.10) is equivalent to*

$$v_+ \in (0, v_-). \tag{3.7}$$

(b) *If $v_- \in (e, f)$, the entropy dissipation D admits three roots: v_- and two other roots denoted by $\varphi_0(v_-) < \psi_0(v_-)$. The entropy condition (2.10) is equivalent to the condition*

$$v_+ \in (0, \alpha] \cup [\beta, \gamma], \tag{3.8}$$

where α is the smallest root, γ is the largest root, and β is the remaining root of D. These values α, β, and γ can be determined as follows.

(i) *If $v_- \in (e, a]$, then*

$$v_- \leq a \leq \varphi^t(v_-) \leq \varphi_0(v_-) < \psi^t(v_-) < \psi_0(v_-).$$

(ii) *If $v_- \in (a, b)$, then*

$$\varphi_0(v_-) < \varphi^t(v_-) < a < v_- < b < \psi^t(v_-) < \psi_0(v_-).$$

(iii) *If $v_- \in [b, f)$, then*

$$\varphi_0(v_-) < \varphi^t(v_-)) < \psi_0(v_-) \leq \psi^t(v_-) \leq b \leq v_-.$$

See Figure 2.

Unlike the boundary functions that make the entropy dissipation vanish in the models in [20, 21], the boundary functions φ_0 and ψ_0 described by Theorem 3.3 are not entirely decreasing. These can be seen by the following theorem.

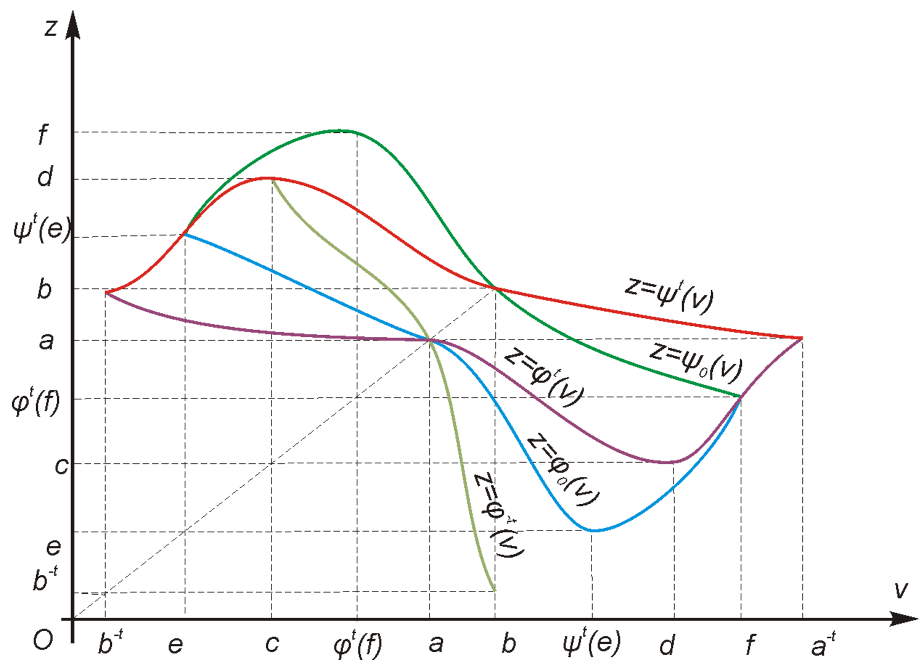

FIGURE 2. The "boundary" functions described by Theorem 3.3

Theorem 3.4. (Corollary 2.3 [19]) *The function* φ_0 *is decreasing in the interval* $[e, \psi^t(e)]$ *with*

$$\varphi_0(\varphi_0(v)) = v, \quad v \in [e, \psi^t(e)],$$

and is increasing in the interval $[\psi^t(e), f]$ *with*

$$\psi_0(\varphi_0(v)) = v, \quad v \in [\psi^t(e), f].$$

The function ψ_0 *is decreasing in the interval* $[\varphi^t(f), f]$ *with*

$$\psi_0(\psi_0(v)) = v, \quad v \in [\varphi^t(f), f],$$

and is increasing in the interval $[e, \varphi^t(f)]$ *with*

$$\varphi_0(\psi_0(v)) = v, \quad v \in [e, \varphi^t(f)].$$

Moreover,

$$\varphi_0(e) = \psi_0(e) = \psi^t(e), \qquad \varphi_0(f) = \psi_0(f) = \varphi^t(f),$$

and

$$\varphi_0(a) = a, \qquad \psi_0(b) = b.$$

Since the boundary functions may serve as the role of a kinetic function, it is derived from Theorem 3.4 that kinetic function may not be globally decreasing or increasing.

The following theorem can be verified easily using Theorem 3.3 and the geometrical observation after (2.6). It characterizes the sets of classical and nonclassical shock waves.

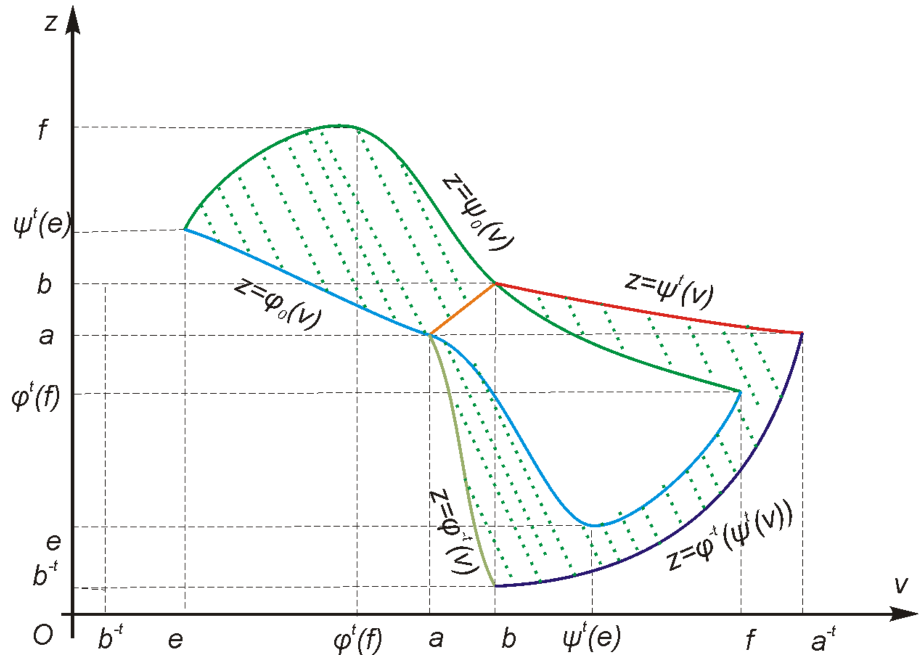

FIGURE 3. Nonclassical shock set given by Theorem 3.5

Theorem 3.5. (Classification of admissible shock waves) *Fix an arbitrary left-hand state* (v_-, u_-)*, and consider the set of right-hand states* (v_+, u_+) *attainable by an admissible* 1*-shock.*

(a) (Classical admissible shock waves) *A classical shock is characterized as follows:*
  (a1) *If* $v_- \in (0, c) \cup (a^{-t}, +\infty)$*, then* $v_+ \in (0, v_-]$.
  (a2) *If* $v_- \in [c, a]$*, then* $v_+ \in (0, v_-] \cup [\varphi^{-t}(v_-), \psi^t(v_-)]$.
  (a3) *If* $v_- \in (a, b)$*, then* $v_+ \in (0, \varphi^{-t}(v_-)] \cup [v_-, \psi^t(v_-)]$.
  (a4) *If* $v_- \in [b, a^{-t}]$*, then* $v_+ \in (0, \varphi^{-t}(\psi^t(v_-))] \cup [\psi^t(v_-), v_-]$.

(b) (Nonclassical admissible shock waves) *A nonclassical shock is characterized as follows:*
  (b1) *If* $v_- \in (e, c]$*, then* $v_+ \in [\varphi_0(v_-), \psi_0(v_-)] := \mathcal{N}(v_-)$.
  (b2) *If* $v_- \in (c, a]$*, then* $v_+ \in [\varphi_0(v_-), \varphi^{-t}(v_-)) \cup (\psi^t(v_-), \psi_0(v_-)] := \mathcal{N}(v_-)$.
  (b3) *If* $v_- \in (a, b)$*, then* $v_+ \in (\varphi^{-t}(v_-), \varphi_0(v_-)] \cup (\psi^t(v_-), \psi_0(v_-)] := \mathcal{N}(v_-)$.
  (b4) *If* $v_- \in [b, f)$*, then* $v_+ \in (\varphi^{-t}(\psi^t(v_-)), \varphi_0(v_-)] \cup [\psi_0(v_-), \psi^t(v_-)) := \mathcal{N}(v_-)$.
  (b5) *If* $v_- \in [f, a^{-t})$*, then* $v_+ \in (\varphi^{-t}(\psi^t(v_-)), \psi^t(v_-)) := \mathcal{N}(v_-)$*. Values in this set also satisfy the Lax shock inequality (2.5).*

See Figure 3.

4. The wave sets and two-parameter family of nonclassical solutions

The Riemann problem (1.1)-(1.3) can be solved in the following manner. We determine the set $\mathcal{W}_1^F(v_L, u_L)$ of all the right-hand states (v, u) that can be reached from the given left-hand state (v_L, u_L) using waves associated with the first characteristic field (1-waves for short), and the set $\mathcal{W}_2^B(v_R, u_R)$ of all the left-hand states (v, u) that can be reached from the given right-hand state (v_R, u_R) using waves associated with the first characteristic field (1-waves for short). Any intermediate state

$$(v_m, u_m) \in \mathcal{W}_1^F(v_L, u_L) \cap \mathcal{W}_2^B(v_R, u_R)$$

determines a Riemann solution of (1.1)-(1.2). Classical wave sets are merely monotone wave curves and there is a unique classical Riemann solution, (see [19], where the construction of the classical wave curves are given). We therefore do not exclude the curves of classical waves in the wave sets $\mathcal{W}_1^F(v_L, u_L)$ and $\mathcal{W}_2^B(v_R, u_R)$, because the inclusion of classical wave curves does not change the geometrical type of these two-parameter wave sets, as seen later on. We still refer to $\mathcal{W}_1^F(v_L, u_L)$ and $\mathcal{W}_2^B(v_R, u_R)$ as nonclassical wave sets. And we construct $\mathcal{W}_1^F(v_L, u_L)$ only, since $\mathcal{W}_2^B(v_R, u_R)$ can be constructed similarly.

Let $\mathcal{N}(v)$ be the set of all values attainable by nonclassical shocks from v, as described in Theorem 3.5. A *kinetic function* θ is thus defined as

$$\begin{aligned} \theta: \quad & [e, a^{-t}] \to \mathbb{R}, \\ & v \mapsto \theta(v) \in \mathcal{N}(v). \end{aligned} \tag{4.1}$$

4.1. Sets of nonclassical waves relying on a kinetic function of the first type.

A kinetic function φ of the first type follows the boundary functions φ^{-t} and φ_0 (see Theorem 3.5) whenever they exist. The monotonicity property of these two functions suggests that there is a value $g \in [b, \psi^t(e)]$ such that φ is decreasing in $[e, g]$ and increasing in $[g, a^{-t}]$. See Figure 4.

The *kinetic relation* of the first type is the requirement that for any nonclassical shock connecting some left-hand state (v_0, u_0) to a right-hand state (v_1, u_1) we have

$$v_1 = \varphi(v_0). \tag{4.2}$$

We are now in a position to construct the wave set $\mathcal{W}_1^F(v_L, u_L)$. First, let $v_L \leq e$. If $v < v_L$, the solution is a classical shock. If $v \in (v_L, a)$, the solution is a rarefaction wave. If $v \in [a, \psi^t(e)]$, there is a unique $v' \in [e, a]$ such that

$$v = \varphi(v'),$$

satisfying the kinetic relation (4.2). The solution is thus a rarefaction wave from v_L to v' followed by a nonclassical shock from v' to v. Any value $v > \psi^t(e)$ is attained by a rarefaction wave from v_L to e followed by a nonclassical shock from e to $\psi^t(e)$, then followed by a rarefaction from $\psi^t(e)$ to v.

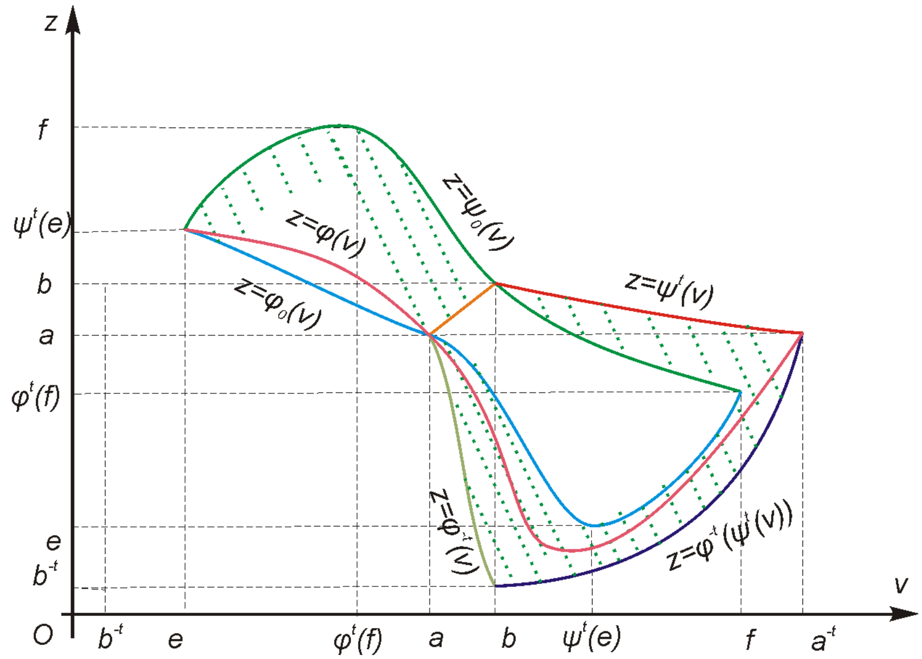

FIGURE 4. Kinetic function φ of the first type

Second, let $v_L \in (e, c)$. If $v < v_L$, the solution is a classical shock. If $v \in (v_L, a]$, the solution is a rarefaction wave. If $v \in (a, \varphi(v_L)]$, there is a unique $v' \in [v_L, a)$ such that

$$v = \varphi(v'),$$

satisfying the kinetic relation (4.2). The solution is thus a rarefaction wave from v_L to v' followed by a nonclassical shock from v' to v. If $v \in (\varphi(v_L), \psi^t(e)]$, there is a unique $v' \in [e, v_L)$ such that

$$v = \varphi(v'),$$

satisfying the kinetic relation (4.2). The solution is thus a classical shock from v_L to v' followed by a nonclassical shock from v' to v. Any value $v > \psi^t(e)$ is attained by a classical shock from v_L to e followed by a nonclassical shock from e to $\psi^t(e)$, then followed by a rarefaction from $\psi^t(e)$ to v.

Third, let $v_L \in [c, a)$. If $v < v_L$, the solution is a classical shock. If $v \in (v_L, a]$, the solution is a rarefaction wave. If $v \in (a, \varphi(v_L)]$, there is a unique $v' \in [v_L, a)$ such that

$$v = \varphi(v'),$$

satisfying the kinetic relation (4.2). The solution is thus a rarefaction wave from v_L to v' followed by a nonclassical shock from v' to v. If $v \in (\varphi(v_L), \psi^t(e)]$, there is a unique $v' \in [e, v_L)$ such that

$$v = \varphi(v'),$$

satisfying the kinetic relation (4.2). The solution is thus a classical shock from v_L to v' followed by a nonclassical shock from v' to v, if $s_1(v_L, v') \le s_1(v', v)$. Otherwise, the solution follows the classical construction. Any value $v > \psi^t(e)$ is attained by a classical shock from v_L to e followed by a nonclassical shock from e to $\psi^t(e)$, then followed by a rarefaction from $\psi^t(e)$ to v, if $s_1(v_L, e) \le s_1(\psi^t(e), v)$. Otherwise, the solution follows the classical construction.

Fourth, let $v_L \in [a, b)$. Any $v < \varphi(g)$ can be attained by the classical construction. If $v \in [\varphi(g), \varphi(v_L))$, there exists a value $v' \in (v_L, g]$ such that $v = \varphi(v')$. The solution can arrive at v by a nonclassical shock from v' proceeded by classical waves from v_L to v': a classical shock if $v' \in (v_L, \psi^t(v_L)]$ and $s_1(v_L, v') \le s_1(v', v)$, or a classical shock from v_L to $\psi^t(v_L)$ followed by a rarefaction wave from $\psi^t(v_L)$ to v' if $p'(v') \le s_1(v', v)$. Of course, the solution follows the classical construction otherwise. If $v \in (\varphi(v_L), a]$, there is some $v' \in [a, v_L)$ such that $v = \varphi(v')$. The solution is thus a classical shock from v_L to v' followed by a nonclassical shock from v' to v.

Since

$$-s_1^2(b, \varphi(b)) > p'(b), \quad -s_1^2(d, \varphi(d)) < p'(d),$$

there is a smallest $h \in (b, d)$ such that

$$-s_1^2(h, \varphi(h)) = p'(h). \tag{4.3}$$

So, fifth, let $v_L \in [b, h]$. Any $v < \varphi(h)$ can be attained by the classical construction. If $v \in [\varphi(h), \varphi(v_L)]$, there exists a value $v' \in [v_L, h]$ such that $v = \varphi(v')$. The solution is a rarefaction wave from v_L to v' followed by a nonclassical shock from v' to v. If $v \in (\varphi(v_L), a]$, there is some $v' \in [a, v_L)$ such that $v = \varphi(v')$. The solution begins with the classical construction from v_L to v' followed by a nonclassical shock from v' to v. Any $v > a$ can be attained using the classical construction.

Sixth, let $v_L \in (h, a^{-t})$. If

$$-s_1^2(v_L, \varphi(v_L)) \le p'(v_L), \tag{4.4}$$

then the construction in the fifth case can be applied. Otherwise, the line segment connecting $(v_L, p(v_L))$ and $(\varphi(v_L), p(\varphi(v_L)))$ cuts the graph of the pressure function at exactly four points at $v_L > v_1 > v_2 > \varphi(v_L)$. If $v < \varphi(v_L)$ such that there exists some $v' \le v_1, v = \varphi(v')$, then the solution can be a classical shock from v_L to v' followed by a nonclassical shock from v' to v. This construction makes sense if $s_1(v_L, v') \le s_1(v', v)$. Otherwise, the construction is classical. Let $v \ge [\varphi(v_L), a]$ and let $v' \in \varphi^{-1}(v)$. If $v' \in [a, \varphi^t(v_L))$, the solution is a classical shock from v_L to $\varphi^t(v_L)$ followed by a rarefaction wave from $\varphi^t(v_L)$, and continued with a nonclassical shock from v' to v. If $v' \in [\varphi^t(v_L), v_L]$, the solution is a classical shock from v_L to v' followed by a nonclassical shock from v' to v if $s_1(v_L, v') \le s_1(v', v)$. Otherwise, the construction is classical. For $v > a$, the solution uses the classical construction.

Seventh, let $v_L > a^{-t}$. Then, the solution follows the classical solution.

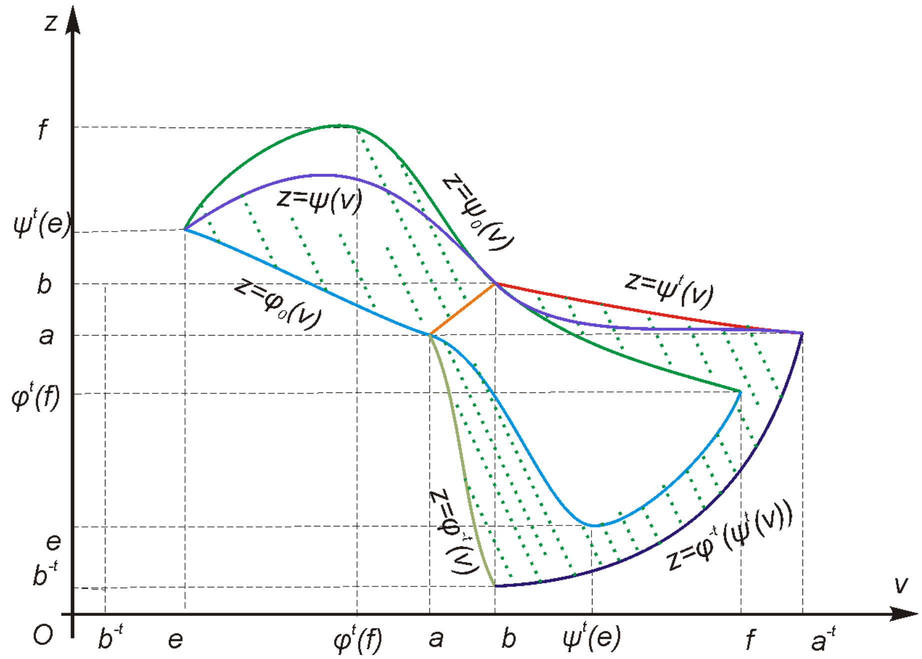

FIGURE 5. Kinetic function ψ of the second type

Remark. Interestingly, we observe a case of multiple solutions as follows. Let $v_L, v_m < a$. The solution can use first a 1-wave from v_L to v_m (a classical shock if $v_m < v_L$ and a rarefaction wave if $v_m \geq v_L$). The solution is continued with a nonclassical solution from v_m to $\varphi(v_m)$, continued by another nonclassical solution from $\varphi(v_m)$ to $\varphi(\varphi(v_m)) = \varphi^2(v_m)$. The solution can be continued to arrive at $\varphi^3(v_m), \varphi^4(v_m), ..., \varphi^n(v_m)$ for any positive integer n. This gives a one-parameter family of solutions. A similar argument can also be made for $v_L > a$. This raises a question for a further study on kinetic relation and conditions for the selection of a unique solution. Nevertheless, one could prevent this by requiring that the kinetic relation is applied at most once to a particular state.

4.2. **Two-dimensional sets of nonclassical waves relying on a kinetic function of the second type.** A kinetic function ψ of the second type follows the boundary functions ψ^t and ψ_0 (see Theorem 3.5) whenever they exist. The monotonicity property of these two functions suggests that there is a value $i \in [c, \varphi^t(f)]$ such that ψ strictly increases for $v \in [e, i]$ and strictly decreases in $[i, a^{-t}]$. See Figure 5.

The *kinetic relation* of the second type is the requirement that for any nonclassical shock connecting some left-hand state (v_0, u_0) to a right-hand state (v_1, u_1) we have

$$v_1 = \psi(v_0). \tag{4.5}$$

Theorem 4.1. *There are two-dimensional sets of waves associated with each characteristic field, which is formed using combinations of classical shocks, rarefaction waves, and nonclassical shocks satisfying the kinetic relation (4.5). Consequently, the Riemann problem (1.1)-(1.2) may admit up to a two-dimensional set of nonclassical solutions.*

Proof. We need only show that there are two-dimensional sets of waves associated with the first family, since the argument for the second family could be similar. Indeed, since

$$-s_1^2(c,\psi(c)) > p'(c), \quad -s_1^2(\varphi^t(f),\psi(\varphi^t(f))) \le p'(\varphi^t(f)), \tag{4.6}$$

there is a smallest value $j \in (c, \varphi^t(f)]$ such that

$$-s_1^2(j,\psi(j)) = p'(j). \tag{4.7}$$

The line segment connecting $(v_1, p(v_1))$ and $(\psi(v_1), p(\psi(v_1)))$ cuts the graph of the pressure at exactly four points at $v_1 < v_2 < v_3 < \psi(v_1)$. We can show that there are several two-dimensional sets of nonclassical waves satisfying the kinetic relation (4.5) as follows.

First, let $v_L \in (0, e)$. From v_L the solution uses a rarefaction wave to some $v_1 \in [e, j]$ followed by a nonclassical shock from v_1 to $\psi(v_1)$, the solution is then continued by a rarefaction wave to reach $v \ge \psi(v_1)$, and by a classical shock to $v < \psi(v_1)$, provided that $s_1(v_1,\psi(v_1)) \le s_1(\psi(v_1), v)$. This makes v_1 as a parameter for the family of wave curves $\mathcal{W}_1^F(v_L, u_L)$, which form a two-dimensional set in the (v, u)-plane.

Second, let $v_L \in [e, j]$. From v_L the solution uses a rarefaction wave to some $v_1 \in [v_L, j]$ or a classical shock from v_L to some $v_1 \in [e, v_L)$, followed by a nonclassical shock from v_1 to $\psi(v_1)$, the solution is then continued by a rarefaction wave to reach $v \ge \psi(v_1)$, and by a classical shock to $v < \psi(v_1)$, provided that $s_1(v_1, \psi(v_1)) \le s_1(\psi(v_1), v)$. This makes v_1 as a parameter for the family of wave curves $\mathcal{W}_1^F(v_L, u_L)$, which form a two-dimensional set in the (v, u)-plane. □

We now continue the construction for nonclassical solutions for $v_L \in (j, b)$. Any state $v \le v_L$ can be reached by a classical solution. Let v_1 be the state at which the secant line segment connecting $(v_L, p(v_L))$ and $(\psi(v_L), p(\psi(v_L)))$ meets the graph of the pressure. Then, any $v \in (v_L, v_1)$ can be reached by a classical shock. If $v \in [v_1, \psi(v_L))$, then the solution is a nonclassical shock from v_L to $\psi(v_L)$ followed by a classical shock from $\psi(v_L)$ to v. If $v \ge \psi(v_L)$, then the solution is a nonclassical shock from v_L to $\psi(v_L)$ followed by a rarefaction wave from $\psi(v_L)$ to v.

Next, assume $v_L \in [b, f]$. Let v_1, v_2 be the states at which the secant line connecting $(v_L, p(v_L))$ and $(\psi(v_L), p(\psi(v_L)))$ meets the graph of the pressure, where $v_2 < \psi(v_L) < v_1 < v_L$. The solution is a nonclassical shock followed by a classical construction from $\psi(v_L)$ to any $v \in [v_2, v_1]$. Other states will be reached using a classical construction.

Finally, let $v_L > f$. Then, the solution uses a classical construction.

Remark. Without the monotonicity assumption on the kinetic function, two-dimensional sets of nonclassical waves satisfying the kinetic relation (4.5) can also be constructed, by a slight modification of the above argument.

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