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STABILITY ANALYSIS FOR LINEAR NON-AUTONOMOUS SYSTEMS WITH CONTINUOUSLY DISTRIBUTED MULTIPLE TIME-VARYING DELAYS AND APPLICATIONS

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Dedicated to Tran Duc Van on the occasion of his sixtieth birthday

Abstract. In this paper, the exponential stability problem of a class of linear non-autonomous systems with continuously distributed multiple time-varying delays is studied. Based on the Lyapunov-Krasovskii functional approach, sufficient conditions for the exponential stability of the system are established via the solution of Riccati differential inequalities. By using this generalized result, new sufficient conditions are derived for the robust stability and stabilization of the systems subjected to uncertainties and external controls. Illustrative numerical examples are given to indicate significant improvements of the results.

1. Introduction

The problems of Lyapunov stability of time-delay systems are of practical and theoretical interest since time delay is often encountered in many industrial and engineering processes [5, 6, 7]. The stability criteria are often developed for linear time-invariant (LTI) systems based on the Lyapunov function method involves the solution of some linear matrix inequalities or algebraic Riccati equations [8, 10, 16, 19, 20]. However, this approach may not be readily applied to linear time-varying (LTV) systems, which are frequently encountered in process dynamics, control, filtering and mobile communication systems. The difficulty is that the solution of a Riccati-type differential inequality is, in general, not uniformly positive definite to be used in a Lyapunov-Krasovskii functional candidate, and hence, the stability analysis becomes more complicated, in particular when the system delay and uncertainties are also time-varying. Moreover, unlike the LTI systems, stability of LTV systems may not be determined by the spectral property of the nominal system matrix as there may exist a stable LTV system with a positive real part of some eigenvalues [11]. In the case of no delay, the definition of stability for LTI

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systems is naturally extended to LTV ones with the use of a suitable time-varying Lyapunov-Krasovskii functional. This leads to conditions for the stability, usually expressed in terms of solutions of differential Riccati-type inequalities [3, 12, 15]. Some existing conditions for the stability for LTV delay systems, obtained in [1, 12, 13, 14], involve a special assumption on the global null-controllability of the nominal system. To the best of our knowledge, there has not been available in the literature a unified approach that addresses the problem of exponential stability for a general class of LTV systems subject to mixed time-varying delays, as dealt with in this paper.

In this paper, we consider LTV systems with time-varying case of continuously distributed multiple delays. By using an improved Lyapunov-Krasovskii functional combined with Riccati equation approach, we propose new criteria for the exponential stability of the system. The delay-dependent conditions are formulated in terms of the solution of Riccati-type differential inequalities, which allow to compute simultaneously the two bounds that characterize the exponential stability rate of the solution. The results are applied to robust stability and stabilization of linear time-varying control systems with mixed delays. Compared to other stability criteria, our result have its own advantages. First, our results can deal with the case of continuously distributed multiple time-varying delays. Second, our approach allows us to apply in robust stability and stabilization of the system subjected to uncertainties and external controls. Therefore, our results extend many related previous ones [3, 8, 10, 13-15, 20].

The paper is organized as follows. Section 2 presents notations, definitions and some auxiliary propositions used in the proof of main results. New delay-dependent sufficient conditions for the exponential stability and applications to robust stability and stabilization are presented in Section 3 and Section 4, respectively. Numerical examples illustrating the obtained results are given in Section 5.

2. Preliminaries

The following notations will be used throughout this paper: $\mathbb{R}^+$ denotes the set of all real nonnegative numbers; $\mathbb{R}^n$ denotes the n-dimensional space with the scalar product $\langle \cdot, \cdot \rangle$ and the vector norm $\| \cdot \|$; A^{T} denotes the transpose of the matrix A, matrix A is symmetric if $A = A^{\mathsf{T}}$; I denotes the identity matrix; $\lambda(A)$ denotes the set of all eigenvalues of A, $\lambda_{\max}(A) = \max\{\mathrm{Re}\,\lambda : \lambda \in \lambda(A)\}$ and $\lambda_{\min}(A) = \min\{\mathrm{Re}\,\lambda : \lambda \in \lambda(A)\}$. Matrix A is called semi-positive definite ($A \geq 0$) if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathbb{R}^n$; A is positive definite ($A > 0$) if $\langle Ax, x \rangle > 0$ for all $x \neq 0$; $A > B$ means $A - B > 0$; M^+ denotes the set of all constant symmetric positive definite matrices; $SM^+[0, \infty)$ denotes the set of all symmetric semi-positive definite matrix functions on $[0, \infty)$; $C([a, b], R^n)$ denotes the set of all continuous functions on $[a, b]$.

Consider a linear non-autonomous system with continuously distributed multiple time-varying delays of the form:

$$
\begin{cases}
\dot{x}(t) = A_0(t)x(t) + \displaystyle\sum_{i=1}^{p} A_i(t)x(t-h_i(t)) \\
\qquad\qquad + \displaystyle\sum_{k=1}^{q} D_k(t)\int_{t-r_k(t)}^{t} x(s)ds, \quad t \in \mathbb{R}^+ \\
x(t) = \phi(t), \quad t \in [-\tau, 0],
\end{cases} \tag{2.1}
$$

where $x(t) \in \mathbb{R}^n$ is the state, $A_i(t), i = 0, 1, \ldots, p,\ D_k(t), k = 1, 2, \ldots, q$ are given continuous matrix functions on $\mathbb{R}^+$, $h_i(t), r_k(t)$ are delay functions which satisfy

$$
\begin{aligned}
&0 \le h_i(t) \le h_i, \quad \dot{h}_i(t) \le \delta_i < 1, \quad i = 1, 2, \ldots, p, \\
&0 \le r_k(t) \le r_k, \quad \dot{r}_k(t) \le \mu_k < 1, \quad k = 1, 2, \ldots, q
\end{aligned} \tag{2.2}
$$

and $\tau = \max\limits_{1\le i\le p, 1\le k\le q}\{h_i, r_k\}$; $\phi(t) \in C([-\tau, 0], \mathbb{R}^n)$ is the initial function with the norm

$$
\|\phi\| = \sup_{t\in[-\tau,0]} \|\phi(t)\|.
$$

Definition 2.1. For given $\alpha > 0$, system (2.1) is said to be $\alpha-$exponentially stable if there exists a number $N > 0$ such that every solution $x(t, \phi)$ of the system satisfies the inequality

$$
\|x(t, \phi)\| \le N\|\phi\|e^{-\alpha t}, \quad t \ge 0.
$$

The following well-known propositions will be used in the proof of our main results.

Proposition 2.1. (Schur complement lemma [2]) *For any matrices X, Y, Z with appropriate dimensions, where $X = X^{\mathsf{T}}, Y = Y^{\mathsf{T}} > 0$, then $X + Z^{\mathsf{T}}Y^{-1}Z < 0$ if and only if*

$$
\begin{bmatrix} X & Z^{\mathsf{T}} \\ Z & -Y \end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix} -Y & Z \\ Z^{\mathsf{T}} & X \end{bmatrix} < 0.
$$

Proposition 2.2. (Matrix Cauchy inequality) *Let $\mathcal{N} \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix. Then for any $x, y \in \mathbb{R}^n$, we have*

$$
2x^{\mathsf{T}}y \le x^{\mathsf{T}}\mathcal{N}x + y\mathcal{N}^{-1}y.
$$

Proposition 2.3. [4] *For any constant matrix $W \in \mathbb{R}^{n\times n}$, $W = W^{\mathsf{T}} > 0$, scalar $\sigma > 0$ and vector function $\omega : [0, \sigma] \to \mathbb{R}^n$ such that the integrals concerned are well defined, then*

$$
\left(\int_0^{\sigma} \omega(s)ds\right)^{\mathsf{T}} W \left(\int_0^{\sigma} \omega(s)ds\right) \le \sigma \int_0^{\sigma} \omega^{\mathsf{T}}(s)W\omega(s)ds.
$$

Proposition 2.4. [6] *Consider the time-delay system*

$$
\dot{x}(t) = f(t, x_t), \quad x(t) = \phi(t), t \in [-h, 0].
$$

If there exist a Lyapunov-Krasovskii functional $V(t, x_t)$ and positive numbers $\lambda_1, \lambda_2, \lambda_3$ such that for every solution $x(t)$ of the system the following conditions hold

(i) $\lambda_1\|x(t)\|^2 \le V(t,x_t) \le \lambda_2\|x_t\|^2$,
(ii) $\dot V(t,x_t) \le -2\lambda_3 V(t,x_t)$,

then the system is exponentially stable, i.e.

$$\exists N>0: \quad \|x(t,\phi)\| \le N\|\phi\|e^{-\lambda_3 t}, \quad t\ge 0.$$

Proposition 2.5. [18] *For any $x,y\in\mathbb{R}^n$ and matrices A,P,E,F,H with appropriate dimensions, $P>0$, $F^{\mathsf T}F\le I$ and scalar $\rho>0$, we have*

(i) $EFH + H^{\mathsf T}F^{\mathsf T}E^{\mathsf T} \le \rho^{-1}EE^{\mathsf T} + \rho H^{\mathsf T}H$;
(ii) *If* $\rho I - HPH^{\mathsf T} > 0$, *then*

$$(A+EFH)P(A+EFH)^{\mathsf T} \le APA^{\mathsf T} + APH^{\mathsf T}(\rho I - HPH^{\mathsf T})^{-1}HPA^{\mathsf T} + \rho^{-1}EE^{\mathsf T}.$$

3. Main results

Given numbers $\alpha>0, \epsilon>0$, $h_i>0, i=1,2,\dots,p$, $r_k>0, k=1,2,\dots,q$, we consider the following Riccati differential inequality (RDI):

$$\dot P_\epsilon(t) + A_0^{\mathsf T}(t)P_\epsilon(t) + P_\epsilon(t)A_0(t) + 2\alpha P_\epsilon(t) + P_\epsilon(t)R(t)P_\epsilon(t) + Q \le 0 \tag{3.1}$$

where

$$P_\epsilon(t) = P(t)+\epsilon I, \quad Q = (p+\sum_{k=1}^{q} r_k)I,$$

$$R(t) = \sum_{i=1}^{p}\frac{e^{2\alpha h_i}}{1-\delta_i}A_i(t)A_i^{\mathsf T}(t) + \sum_{k=1}^{q}\frac{r_k e^{2\alpha r_k}}{1-\mu_k}D_k(t)D_k^{\mathsf T}(t).$$

Denote

$$p_0 = \lambda_{\max}(P(0)), \quad \epsilon_1 = p_0 + \epsilon + \sum_{i=1}^{p}\frac{1-e^{-2\alpha h_i}}{2\alpha} + \sum_{k=1}^{q}\frac{e^{-2\alpha r_k}+2\alpha r_k - 1}{4\alpha^2}.$$

We have the following result for the $\alpha-$exponential stability of system (2.1).

Theorem 3.1. *For given $\alpha>0$, system (2.1) is $\alpha-$exponentially stable if there exist $\epsilon>0$ and $P\in SM^+[0,\infty)$ such that the RDI (3.1) holds. Moreover, the solution $x(t,\phi)$ of system (2.1) satisfies the following exponential condition*

$$\|x(t,\phi)\| \le N\|\phi\|e^{-\alpha t}, \quad t\ge 0,$$

where, $N = \sqrt{\dfrac{\epsilon_1}{\epsilon}}$.

Proof. Let $\epsilon>0$ and $P\in SM^+[0,\infty)$ be a solution of RDI (3.1) . Consider the following Lyapunov-Krasovskii functional

$$V(t,x_t) = V_1+V_2+V_3+V_4, \quad t\ge 0,$$

where

$$
\begin{aligned}
&V_1 = \langle P(t)x(t), x(t)\rangle \\
&V_2 = \epsilon\|x(t)\|^2 \\
&V_3 = \sum_{i=1}^{p} \int\limits_{t-h_i(t)}^{t} e^{2\alpha(s-t)}\|x(s)\|^2 ds \\
&V_4 = \sum_{k=1}^{q} \int\limits_{t-r_k(t)}^{t} \int\limits_{s}^{t} e^{2\alpha(\xi-t)}\|x(\xi)\|^2 d\xi ds.
\end{aligned} \tag{3.2}
$$

It is easy to see that the function $V(t, x_t)$ is positive definite and

$$
\epsilon\|x(t)\|^2 \le V(t, x_t), \quad t \in \mathbb{R}^+. \tag{3.3}
$$

Taking the derivative of V_1 and V_2 respectively along the solution of system (2.1), we have

$$
\begin{aligned}
\dot V_1 &= \langle \dot P(t)x(t), x(t)\rangle + 2\langle P(t)\dot x(t), x(t)\rangle \\
&= \langle (\dot P(t) + A_0^{\mathsf T}(t)P(t) + P(t)A_0(t))x(t), x(t)\rangle \\
&\quad + 2\sum_{i=1}^{p}\langle P(t)A_i(t)x(t-h_i(t)), x(t)\rangle \\
&\quad + 2\sum_{k=1}^{q}\langle P(t)D_k(t)\int_{t-r_k(t)}^{t} x(s)ds, x(t)\rangle \\
\dot V_2(t) &= 2\epsilon\langle (A_0(t)x(t), x(t)\rangle \\
&\quad + 2\epsilon\sum_{i=1}^{p}\langle A_i(t)x(t-h_i(t)), x(t)\rangle \\
&\quad + 2\epsilon\sum_{k=1}^{q}\langle D_k(t)\int_{t-r_k(t)}^{t} x(s)ds, x(t)\rangle.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\dot V_1 + \dot V_2 &= \langle (\dot P_\epsilon(t) + A_0^{\mathsf T}(t)P_\epsilon(t) + P_\epsilon(t)A_0(t))x(t), x(t)\rangle \\
&\quad + 2\sum_{i=1}^{p}\langle P_\epsilon(t)A_i(t)x(t-h_i(t)), x(t)\rangle \\
&\quad + 2\sum_{k=1}^{q}\langle P_\epsilon(t)D_k(t)\int_{t-r_k(t)}^{t} x(s)ds, x(t)\rangle.
\end{aligned} \tag{3.4}
$$

Using Propositions 2.2 and 2.3, we have the following estimates

$$
2\langle P_\epsilon(t)A_i(t)x(t-h_i(t)), x(t)\rangle \le \frac{e^{2\alpha h_i}}{1-\delta_i}\langle P_\epsilon(t)A_i(t)A_i^{\mathsf T}(t)P_\epsilon(t)x(t), x(t)\rangle
$$

$$+ (1-\delta_i)e^{-2\alpha h_i}\|x(t-h_i(t))\|^2, \quad 1 \le i \le p.$$

$$\begin{aligned}
2\langle P_\epsilon(t)D_k(t)\int_{t-r_k(t)}^t x(s)ds, x(t)\rangle &\le \frac{r_k e^{2\alpha r_k}}{1-\mu_k}\langle P_\epsilon(t)D_k(t)D_k^{\mathsf{T}}(t)x(t), x(t)\rangle \\
&\quad + \frac{1}{r_k}(1-\mu_k)e^{-2\alpha r_k}\left(\int_{t-r_k(t)}^t x(s)ds\right)^{\mathsf{T}}\left(\int_{t-r_k(t)}^t x(s)ds\right) \\
&\le \frac{r_k e^{2\alpha r_k}}{1-\mu_k}\langle P_\epsilon(t)D_k(t)D_k^{\mathsf{T}}(t)x(t), x(t)\rangle \\
&\quad + (1-\mu_k)e^{-2\alpha r_k}\int_{t-r_k(t)}^t \|x(s)\|^2 ds, \quad 1 \le k \le q.
\end{aligned}$$

Therefore, from (3.4) we have

$$\begin{aligned}
\dot V_1 + \dot V_2 &\le \langle(\dot P_\epsilon(t) + P_\epsilon(t)A_0(t) + A_0^{\mathsf{T}}(t)P_\epsilon(t) + P_\epsilon(t)R(t)P_\epsilon(t))x(t), x(t)\rangle \\
&\quad + \sum_{i=1}^p (1-\delta_i)e^{-2\alpha h_i}\|x(t-h_i(t))\|^2 \\
&\quad + \sum_{k=1}^q (1-\mu_k)e^{-2\alpha r_k}\int_{t-r_k(t)}^t \|x(s)\|^2 ds.
\end{aligned} \tag{3.5}$$

Next, the derivative of V_3 and V_4 along the solution of system (2.1) are given by

$$\begin{aligned}
\dot V_3 &= \sum_{i=1}^p \|x(t)\|^2 - \sum_{i=1}^p (1-\dot h_i(t))e^{-2\alpha h_i(t)}\|x(t-h_i(t))\|^2 - 2\alpha V_3 \\
&\le p\|x(t)\|^2 - \sum_{i=1}^p (1-\delta_i)e^{-2\alpha h_i}\|x(t-h_i(t))\|^2 - 2\alpha V_3.
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
\dot V_4 &= \sum_{k=1}^q r_k(t)\|x(t)\|^2 - \sum_{k=1}^q (1-\dot r_k(t))\int_{t-r_k(t)}^t e^{2\alpha(s-t)}\|x(s)\|^2 ds - 2\alpha V_4 \\
&\le \sum_{k=1}^q r_k\|x(t)\|^2 - \sum_{k=1}^q (1-\mu_k)e^{-2\alpha r_k}\int_{t-r_k(t)}^t \|x(s)\|^2 ds - 2\alpha V_4.
\end{aligned} \tag{3.7}$$

Combining (3.4)-(3.7) gives

$$\begin{aligned}
\dot V(t, x_t) + 2\alpha V(t, x_t) \le \langle(\dot P_\epsilon(t) &+ A_0^{\mathsf{T}}(t)P_\epsilon(t) + P_\epsilon(t)A_0(t) + 2\alpha P_\epsilon(t) \\
&+ P_\epsilon(t)R(t)P_\epsilon(t) + Q)x(t), x(t)\rangle.
\end{aligned} \tag{3.8}$$

Then

$$\dot V(t, x_t) + 2\alpha V(t, x_t) \le 0, \quad t \ge 0. \tag{3.9}$$

Therefore, by Proposition 2.4, the system is $\alpha-$ exponential stable. To find the stability factor, integrating both sides of (3.9) from 0 to t gives

$$V(t, x_t) \leq V(0, x_0)e^{-2\alpha t}, \quad t \in \mathbb{R}^+.$$

Taking the estimation (3.3) into account, we finally obtain

$$\|x(t, \phi)\| \leq \sqrt{\frac{V(0, x_0)}{\epsilon}} e^{-\alpha t}, \quad t \geq 0.$$

We now estimate the value $V(0, x_0)$ as follows.

$$\begin{aligned} V(0, x_0) &= \langle P(0)x(0), x(0)\rangle + \epsilon\|x(0)\|^2 \\ &\quad + \sum_{i=1}^{p} \int_{-h_i(0)}^{0} e^{2\alpha s}\|x(s)\|^2 ds + \sum_{k=1}^{q} \int_{-r_k(0)}^{0} \int_{s}^{0} e^{2\alpha\xi}\|x(\xi)\|^2 d\xi ds \\ &\leq (p_0 + \epsilon)\|\phi\|^2 + \Big[\sum_{i=1}^{p} \int_{-h_i}^{0} e^{2\alpha s} ds + \sum_{k=1}^{q} \int_{-r_k}^{0} \int_{s}^{0} e^{2\alpha\xi} d\xi ds\Big]\|\phi\|^2 \\ &= (p_0 + \epsilon)\|\phi\|^2 + \Big[\sum_{i=1}^{p} \frac{1 - e^{-2\alpha h_i}}{2\alpha} + \sum_{k=1}^{q} \frac{e^{-2\alpha r_k} + 2\alpha r_k - 1}{4\alpha^2}\Big]\|\phi\|^2 \\ &= \epsilon_1\|\phi\|^2, \end{aligned}$$

where

$$\epsilon_1 = p_0 + \epsilon + \sum_{i=1}^{p} \frac{1 - e^{-2\alpha h_i}}{2\alpha} + \sum_{k=1}^{q} \frac{e^{-2\alpha r_k} + 2\alpha r_k - 1}{4\alpha^2}. \tag{3.10}$$

Hence

$$\|x(t, \phi)\| \leq N\|\phi\|e^{-\alpha t}, \quad t \geq 0,$$

where $N = \sqrt{\frac{\epsilon_1}{\epsilon}}$. This completes the proof of the theorem. □

Remark 3.2. Theorem 3.1 provides sufficient conditions for the exponential stability, which includes the results of [10, 14] as special cases.

Remark 3.3. The exponential stability conditions are given in terms of the solution of RDIs. Various efficient methods for solving RDIs can be found in [9, 17].

4. Applications

In this section, we apply the obtained results to the robust stability and stabilization of uncertain linear control systems with discrete and distributed multiple delays .

4.1. **Robust stability.** Consider the following uncertain LTV system with mixed multiple delays

$$(4.1)\qquad \begin{cases} \dot{x}(t) = (A_0 + \Delta A_0(t))x(t) + \sum_{i=1}^{p}(A_i + \Delta A_i(t))x(t - h_i(t)) \\ \qquad + \sum_{k=1}^{q}(D_k + \Delta D_k(t)) \int_{t-r_k(t)}^{t} x(s)ds, \quad t \in \mathbb{R}^+ \\ x(0) = \phi(t), \quad t \in [-\tau, 0], \end{cases}$$

where $h_i(t), r_k(t)$ are time-varying delay functions satisfying (2.2) and A_0, A_i, D_k, $1 \le i \le p$, $1 \le k \le q$ are given constant matrices and $\Delta A_0(t)$, $\Delta A_i(t)$, $\Delta D_k(t)$ are uncertainties of the form

$$\Delta A_0(t) = E_0 F_0(t) H_0, \quad \Delta A_i(t) = E_i^a F_i^a(t) H_i^a, \quad \Delta D_k(t) = E_k^d F_k^d(t) H_k^d,$$

where $E_0, H_0, E_i^a, H_i^a, 1 \le i \le p$ and $E_k^d, H_k^d, 1 \le k \le q$ are given real matrices and $F_0(t), F_i^a(t), 1 \le i \le p, F_k^d(t), 1 \le k \le q$ are uncertainties satisfying the following conditions

$$F_0^{\top}(t)F_0(t) \le I,\ F_i^{a\top}(t)F_i^a(t) \le I,\ F_k^{d\top}(t)F_k^d(t) \le I,\ i = 1, 2, ..., p,\ k = 1, 2, ..., q.$$

Then, we have the following result for the $\alpha-$exponentially stable of the system (4.1).

Theorem 4.1. *For given $\alpha > 0$, system (4.1) is $\alpha-$exponentially stable if there exist a symmetric positive definite matrix P and positive numbers $\rho_0, \rho_i, \nu_k, 1 \le i \le p, 1 \le k \le q$ such that $\rho_i I - H_i^a H_i^{a\top} > 0, \nu_k I - H_k^d H_k^{d\top} > 0$ and the following LMI holds:*

$$(4.2)\qquad \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & \Omega_{15} & \Omega_{16} & \Omega_{17} \\ * & -\Omega_{22} & 0 & 0 & 0 & 0 & 0 \\ * & * & -\Omega_{33} & 0 & 0 & 0 & 0 \\ * & * & * & -\Omega_{44} & 0 & 0 & 0 \\ * & * & * & * & -\Omega_{55} & 0 & 0 \\ * & * & * & * & * & -\Omega_{66} & 0 \\ * & * & * & * & * & * & -\Omega_{77} \end{bmatrix} < 0$$

where

$$\Omega_{11} = A_0^{\top} P + PA_0 + 2\alpha P + \rho_0 H_0^{\top} H_0 + \Big(p + \sum_{k=1}^{q} r_k\Big)I,$$

$$\Omega_{12} = \begin{bmatrix} PA_1 H_1^{a\top} & \dots & PA_p H_p^{a\top} \end{bmatrix},$$

$$\Omega_{13} = \begin{bmatrix} PD_1 H_1^{d\top} & \dots & PD_q H_q^{d\top} \end{bmatrix},$$

$$\Omega_{14} = \begin{bmatrix} PE_0 & PE_1^a & \dots & PE_p^a \end{bmatrix},$$

$$\Omega_{15} = \begin{bmatrix} PE_1^d & PE_2^d & \dots & PE_q^d \end{bmatrix},$$

$$\Omega_{16} = \begin{bmatrix} PA_1 & PA_2 & \dots & PA_p \end{bmatrix},$$

$$\Omega_{17} = \begin{bmatrix} PD_1 & PD_2 & \dots & PD_q \end{bmatrix},$$

$$\Omega_{22} = \mathrm{diag}\Big\{(1-\delta_1)e^{-2\alpha h_1}(\rho_1 I - H_1^a H_1^{a\top}), \dots, (1-\delta_p)e^{-2\alpha h_p}(\rho_p I - H_p^a H_p^{a\top})\Big\},$$
$$\Omega_{33} = \mathrm{diag}\Big\{(1-\mu_1)r_1^{-1}e^{-2\alpha r_1}(\nu_1 I - H_1^d H_1^{d\top}), \dots, (1-\mu_q)r_q^{-1}e^{-2\alpha r_q}(\nu_q I - H_q^d H_q^{d\top})\Big\},$$
$$\Omega_{44} = \mathrm{diag}\Big\{\rho_0 I, (1-\delta_1)e^{-2\alpha h_1}\rho_1 I, \dots, (1-\delta_p)e^{-2\alpha h_p}\rho_p I\Big\},$$
$$\Omega_{55} = \mathrm{diag}\Big\{(1-\mu_1)r_1^{-1}e^{-2\alpha r_1}\nu_1 I, \dots, (1-\mu_q)r_q^{-1}e^{-2\alpha r_q}\nu_q I\Big\},$$
$$\Omega_{66} = \mathrm{diag}\Big\{(1-\delta_1)e^{-2\alpha h_1} I, \dots, (1-\delta_p)e^{-2\alpha h_p} I\Big\},$$
$$\Omega_{77} = \mathrm{diag}\Big\{(1-\mu_1)r_1^{-1}e^{-2\alpha r_1} I, \dots, (1-\mu_q)r_q^{-1}e^{-2\alpha r_q} I\Big\}.$$

Proof. Let us denote

$$A_0(t) = A_0 + \Delta A_0(t), A_i(t) = A_i + \Delta A_i(t), D_k(t) = D_k + \Delta D_k(t).$$

Using Proposition 2.5, we have

$$PE_0F_0(t)H_0 + H_0^\top F_0^\top(t)E_0^\top P \le \rho_0^{-1}PE_0E_0^\top P + \rho_0 H_0^\top H_0,$$
$$A_i(t)A_i^\top(t) \le A_iA_i^\top + A_iH_i^{a\top}\Big(\rho_i I - H_i^a H_i^{a\top}\Big)^{-1}H_i^aA_i^\top + \rho_i^{-1}E_i^aE_i^{aT},$$
$$D_k(t)D_k^\top(t) \le D_kD_k^\top + D_kH_k^{dT}\Big(\nu_k I - H_k^d H_k^{d\top}\Big)^{-1}H_k^dD_k^\top + \nu_k^{-1}E_k^dE_k^{d\top},$$
$$1 \le i \le p, \quad 1 \le k \le q.$$

Applying Theorem 3.1 to the case $P_\epsilon(t) = P$, we reduce the RDI (3.1) to

$$A_0^\top(t)P + PA_0(t) + 2\alpha P + PR(t)P + (p + \sum_{k=1}^q r_k)I \le 0.$$

We have

$$\begin{aligned}
&A_0^\top(t)P + PA_0(t) + 2\alpha P + PR(t)P + (p + \sum_{k=1}^q r_k)I \\
&\quad \le A_0^\top P + PA_0 + 2\alpha P + Q + \rho_0^{-1}PE_0^aE_0^{a\top}P + \rho_0 H_0^{a\top}H_0^a \\
&\qquad + \sum_{i=1}^p \frac{e^{2\alpha h_i}}{1-\delta_i}PA_iA_i^\top P + \sum_{i=1}^p \frac{e^{2\alpha h_i}}{1-\delta_i}\rho_i^{-1}PE_i^aE_i^{a\top}P \\
&\qquad + \sum_{i=1}^p \frac{e^{2\alpha h_i}}{1-\delta_i}PA_iH_i^{a\top}\Big(\rho_i I - H_i^aH_i^{a\top}\Big)^{-1}H_i^aA_i^\top P \\
&\qquad + \sum_{k=1}^q \frac{r_ke^{2\alpha r_k}}{1-\mu_k}PD_kD_k^\top P + \sum_{k=1}^q \frac{r_ke^{2\alpha r_k}}{1-\mu_k}\nu_k^{-1}PE_k^dE_k^{d\top}P \\
&\qquad + \sum_{k=1}^q \frac{r_ke^{2\alpha r_k}}{1-\mu_k}PD_kH_k^{d\top}\Big(\nu_k I - H_k^dH_k^{d\top}\Big)^{-1}H_k^dD_k^\top P.
\end{aligned} \tag{4.3}$$

Using Schur complement lemma (Proposition 2.1) and from (4.2), (4.3) we obtain the similar estimation (3.9) for system (4.1) and hence the $\alpha-$exponential stability of the system (4.1) is derived. Moreover, in this case, the stability factor is given by $N = \sqrt{\frac{\lambda_2}{\lambda_1}}$, where $\lambda_1 = \lambda_{\min}(P),\ \lambda_2 = \lambda_{\max}(P) + \sum_{i=1}^p \frac{1-e^{-2\alpha h_i}}{2\alpha} + \sum_{k=1}^q \frac{e^{-2\alpha r_k} + 2\alpha r_k - 1}{4\alpha^2}$. The proof is complete. □

Remark 4.2. The robust stability conditions are obtained in terms of LMIs, which can be easily determined by utilizing MATLABs LMI Control Toolbox [2]. Moreover, it is worth noting that the results on the asymptotic stability of uncertain time-delay systems obtained in [3, 8, 20] are derived from Theorem 4.1 as a special case.

4.2. **Stabilization.** Consider the following linear time-varying control system with mixed multiple delays

$$\begin{cases} \dot{x}(t) = A_0(t)x(t) + \sum_{i=1}^p A_i(t)x(t-h_i(t)) + B(t)u(t) \\ \qquad\qquad + \sum_{k=1}^q D_k(t)\int_{t-r_k(t)}^t x(s)ds, \quad t \in \mathbb{R}^+, \\ x(t) = \phi(t), \quad t \in [-\tau, 0], \end{cases} \tag{4.4}$$

where $u(t) \in \mathbb{R}^m$ is the control, $A_0(t), A_i(t), D_k(t), 1 \le i \le p, 1 \le k \le q$ and $B(t)$ are given continuous matrix functions on $[0, \infty)$, $h_i(t), r_k(t)$ are delay functions satisfying (2.2).

We recall that system (4.4) is $\alpha-$exponentially stabilizable if there is a feedback control $u(t) = K(t)x(t)$ such that the closed-loop system

$$\begin{cases} \dot{x}(t) = [A_0(t) + B(t)K(t)]x(t) + \sum_{i=1}^p A_i(t)x(t-h_i(t)) \\ \qquad\qquad + \sum_{k=1}^q D_k(t)\int_{t-r_k(t)}^t x(s)ds, \quad t \in \mathbb{R}^+, \\ x(t) = \phi(t), \quad t \in [-\tau, 0] \end{cases} \tag{4.5}$$

is $\alpha-$exponentially stable.

For numbers $\alpha > 0, \epsilon > 0,\ h_i, r_k > 0,\ 1 \le i \le p, 1 \le k \le q$, we denote

$$P_\epsilon(t) = P(t) + \epsilon I, \quad Q = (p + \sum_{k=1}^q r_k)I,$$

$$R(t) = \sum_{i=1}^p \frac{e^{2\alpha h_i}}{1-\delta_i} A_i(t)A_i^{\mathsf{T}}(t) + \sum_{k=1}^q \frac{r_k e^{2\alpha r_k}}{1-\mu_k} D_k(t)D_k^{\mathsf{T}}(t),$$

$$\widehat{R}(t) = \frac{1}{2}\Big[B(t)B^{\mathsf{T}}(t) - I\Big]R(t) + \frac{1}{2}R(t)\Big[B(t)B^{\mathsf{T}}(t) - I\Big],$$

and consider the following RDI

$$\dot{P}_\epsilon(t) + A_0^{\mathsf{T}}(t)P_\epsilon(t) + P_\epsilon(t)A_0(t) + 2\alpha P_\epsilon(t) - P(t)\widehat{R}(t)P(t) + Q \le 0. \tag{4.6}$$

Theorem 4.3. *For given $\alpha > 0$, system (4.4) is $\alpha-$exponentially stabilizable if there exist $\epsilon > 0$ and $P \in SM^+[0,\infty)$ such that the RDI (4.6) holds. The feedback stabilizing control is given by*

$$u(t) = -\frac{1}{2}B^{\mathsf{T}}(t)R(t)P_\epsilon(t)x(t), \quad t \ge 0.$$

Proof. With the feedback control $u(t) = K(t)x(t)$, where $K(t) = -\frac{1}{2}B^{\mathsf{T}}(t)R(t)P_\epsilon(t)$, the closed-loop system of (4.4) is given by

$$\dot{x}(t) = \Big[A_0(t) + B(t)K(t)\Big]x(t) + \sum_{i=1}^{p} A_i(t)x(t-h_i(t)) + \sum_{k=1}^{q} D_k(t)\int_{t-r_k(t)}^{t} x(s)ds. \tag{4.7}$$

Denote $\overline{A}_0(t) = A_0(t) + B(t)K(t)$, then system (4.7) can be written as

$$\dot{x}(t) = \overline{A}_0(t)x(t) + \sum_{i=1}^{p} A_i(t)x(t-h_i(t)) + \sum_{k=1}^{q} D_k(t)\int_{t-r_k(t)}^{t} x(s)ds.$$

From (4.6), we have

$$\begin{aligned} P_\epsilon(t)\widehat{R}(t)P_\epsilon(t) &= \frac{1}{2}P_\epsilon(t)[B(t)B^{\mathsf{T}}(t) - I]R(t)P_\epsilon(t) + \frac{1}{2}P_\epsilon(t)R(t)[B(t)B^{\mathsf{T}}(t) - I]P_\epsilon(t) \\ &= -\Big[P_\epsilon(t)B(t)K(t) + K^{\mathsf{T}}(t)B^{\mathsf{T}}(t)P_\epsilon(t)\Big] - P_\epsilon(t)R(t)P_\epsilon(t). \end{aligned}$$

It follows that

$$\begin{aligned} A_0^{\mathsf{T}}(t)P_\epsilon(t) + P_\epsilon(t)A_0(t) - P_\epsilon(t)\widehat{R}(t)P_\epsilon(t) &= \Big[A_0^{\mathsf{T}}(t) + K^{\mathsf{T}}(t)B^{\mathsf{T}}(t)\Big]P_\epsilon(t) \\ &\quad + P_\epsilon(t)\Big[A_0(t) + B(t)K(t)\Big] + P(t)R(t)P(t) \\ &= \overline{A}_0^{\mathsf{T}}(t)P_\epsilon(t) + P_\epsilon(t)\overline{A}_0(t) + P_\epsilon(t)R(t)P_\epsilon(t). \end{aligned}$$

Therefore,

$$\dot{P}_\epsilon(t) + \overline{A}_0^{\mathsf{T}}(t)P_\epsilon(t) + P_\epsilon(t)\overline{A}_0(t) + 2\alpha P_\epsilon(t) + P_\epsilon(t)R(t)P_\epsilon(t) + Q \le 0.$$

Applying Theorem 3.1, we conclude that the closed-loop system (4.7) is $\alpha-$exponentially stable. The proof is complete. □

Remark 4.4. Theorem 4.2 provides sufficient conditions for the exponential stabilization, which includes the results in [13, 14, 15].

5. Examples

Example 5.1. (Exponential stability) Consider a linear time-varying system with discrete and distributed delays of the form

$$\dot{x}(t) = A_0(t)x(t)+A_1(t)x(t-h_1(t))+A_2(t)x(t-h_2(t))+D(t)\int_{t-r(t)}^{t} x(s)ds, \tag{5.1}$$

where

$$A_0(t) = \begin{bmatrix} a_0(t) & b_0(t) \\ b_0(t) & c_0(t) \end{bmatrix}, \quad A_1(t) = \frac{1}{2}e^{-1+t}\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix},$$

$$A_2(t) = \frac{1}{2}e^{-1+t}\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \quad D(t) = e^{t-0.5}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

$$a_0(t) = -3(1+e^{2t}) - \frac{5e^{2t}}{2(1+e^{2t})}, \quad b_0(t) = 1+e^{2t},$$

$$c_0(t) = -\frac{3}{2}(1+e^{2t}) - \frac{5e^{2t}}{2(1+e^{2t})}$$

and $h_1(t) = \sin^2\frac{t}{2}, \quad h_2(t) = \cos^2\frac{t}{2}, \quad r(t) = 1/2\cos^2\frac{3t}{2}$.

Note that we cannot apply the approach used in [8, 19, 20] because their conditions lead to an unsolvable infinite system of LMIs. Indeed, the Lyapunov-Krasovskii functional used, for example, in [19] leads to the LMI conditions in Theorem 3.1 therein, which is of a function of the time t and hence to solve a system of time-varying LMIs of the form

$$\Psi[A_0(t), A_1(t), P, Q, R] < 0 \quad \forall t \geq 0.$$

Clearly, the Matlab LMI toolbox cannot be applied to solve this time-varying LMI with respect to the positive definite matrix solutions P, Q, R. However, we can find the solution of RDI (3.1) associated to system (5.1) . We have $h_1 = h_2 = 1, r = 0.5$ and $\delta_1 = \delta_2 = 0.5$, $\mu_1 = 0.75$. Taking $\alpha = 1$, $\epsilon = 1$, then we can verify that

$$P(t) = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$$

is a solution of RDI (3.1). By Theorem 3.1, the system (5.1) is exponentially stable with decay rate $\alpha = 1$. By simple computation, we obtain $p_0 = 1, \epsilon_1 = 3 - e^{-2} + \frac{1}{4}e^{-1}$. From (3.10), we have the stability factor $N = \sqrt{\epsilon_1} \simeq 1.7195$ and the solution of system of (5.1) satisfies

$$\|x(t,\phi)\| \leq 1.72e^{-t}\|\phi\|, \quad t \geq 0.$$

Example 5.2. (Stabilization) Consider control system (4.4), where $p = q = 1$ and

$$A_0(t) = \begin{bmatrix} -\frac{3}{2(1+e^{-t})} & \frac{3}{2}e^{-t}(1+e^{-t}) \\ \frac{3}{2}e^{-t}(1+e^{-t}) & -\frac{3}{2(1+e^{-t})} \end{bmatrix}, \quad A_1(t) = \frac{1}{2}e^{0.5t-1}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},$$

$$D(t) = \frac{1}{2}e^{0.5(t-1)} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B(t) = \sqrt{1+e^{-t}} \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$h(t) = 2\sin^2 \frac{3}{8}t, \quad r(t) = \cos^2 \frac{3}{4}t.$$

We have $h = 2, r = 1$ and $\delta_1 = \mu_1 = 0.75$. Taking $\alpha = 0.5, \epsilon = 1$, then we verify that

$$P(t) = e^{-t} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is a solutions of RDI (4.6). By Theorem 4.2, system (4.4) is 0.5−exponentially stabilizable. The feedback controller is given by

$$u(t) = -e^{-t}(1+e^{-t})^{\frac{3}{2}} \begin{bmatrix} 2 & 1 \end{bmatrix} x(t), \quad t \geq 0.$$

Moreover, the solution of the closed-loop system satisfies

$$\|x(t,\phi)\| \leq 1.798e^{-0.5t}\|\phi\|, \quad t \geq 0.$$

Example 5.3. (Robust stability) Consider system (4.1), where $p = q = 2$ and

$$A_0 = \begin{bmatrix} -10 & 0 \\ 1 & -12 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0.5 \\ 1 & 1 \end{bmatrix},$$
$$D_1 = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 0.5 \\ 0 & -1 \end{bmatrix},$$
$$E_0 = E_i^a = E_i^d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad H_0 = H_i^a = H_i^d = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \; i = 1, 2,$$
$$h_1(t) = 0.5\sin^2 t, \; h_2(t) = 0.5\cos^2 t, \; r_1(t) = 0.2\sin^2 2.5t, \; r_2(t) = 0.2\cos^2 2.5t.$$

We have, $h_1 = h_2 = 0.5, r_1 = r_2 = 0.2, \delta_1 = \delta_2 = 0.5$ and $\mu_1 = \mu_2 = 0.5$. For given $\alpha = 0.5$, by using LMI Matlab Toolbox, we find that all conditions in Theorem 4.1 are feasible with $\rho_0 = 10, \rho_i = 10, \nu_i = 10, i = 1, 2$ and

$$P = \begin{bmatrix} 1.2528 & -0.9050 \\ -0.9050 & 1.3518 \end{bmatrix}.$$

By Theorem 4.1, system (4.1) is robust exponentially stable with decay rate $\alpha = 0.5$. Moreover, every solution $x(t,\phi)$ of the system satisfies the following inequality

$$\|x(t,\phi)\| \leq 2.768\|\phi\|e^{-0.5t}, \quad t \geq 0.$$

Conclusions

In this paper, new sufficient conditions for the exponential stability of LTV systems with continuously distributed multiple time-varying delays have been established by employing an augmented Lyapunov-Krasovskii functional. The conditions are formulated in terms of the solution of certain Riccati differential inequalities, which allow us to compute simultaneously the two bounds that characterize the exponential stability rate of the solution. The results have been applied to obtain new conditions for robust stability and stabilization of uncertain linear time-varying control systems with mixed multiple delays. Illustrative numerical examples are given to indicate significant improvements and the application of the results.

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