ACTA MATHEMATICA VIETNAMICA
Volume 36, Number 2, 2011, pp. 397–417

ASYMPTOTIC PROPERTIES OF SOLUTIONS OF OPERATOR EQUATIONS

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Dedicated to Tran Duc Van on the occasion of his sixtieth birthday

Abstract. We study properties of solutions of operator equations $\mathcal{D}u-\mathcal{B}u = f$ $(*)$ where $\mathcal{D}$ is the generator of an isometric group $V(t)$ on a Banach space $\mathcal{F}$ and $\mathcal{B}$ is a closed operator commuting with $\mathcal{D}$. We introduce the equation spectrum Σ and prove that if f is an almost periodic element (with respect to the group $V(t)$) and Σ is countable, then any solution u of (*) is almost periodic, provided either $\mathcal{F} \not\supset c_0$ or u is totally ergodic. The presented approach when applied to functional-differential equations gives spectral criteria of almost periodicity of bounded uniformly continuous solutions. The discrete version of the results, with applications to properties of solutions of functional-difference equations, also is described.

1. Introduction

Let E be a Banach space and $BUC(\mathbb{R}, E)$ the space of uniformly continuous bounded functions on $\mathbb{R}$ with values in E, with sup-norm. A function $f \in BUC(\mathbb{R}, E)$ is called *almost periodic*, if the family of translates $\{f_t(s) := f(s+t) : t \in \mathbb{R}\}$ is relatively compact in $BUC(\mathbb{R}, E)$. The classical Loomis theorem, which states that, for a scalar function, countability of its spectrum implies almost periodicity, has been generalized to vector-valued functions (see, e.g. [4, 14, 21]). This generalized Loomis theorem plays an important role in investigations of almost periodicity of solutions of various classes of linear time-invariant differential and functional-differential equations in Banach spaces (see e.g. [1,4,12,14,18,21,23–25]). Typically, the main condition imposed on equations considered in these papers is the countability of some spectral set associated with the equation, which implies the countability of the spectrum of the solution, and, therefore, almost periodicity follows from the Loomis theorem.

In this paper, we consider the abstract operator equations of the following form $\mathcal{L}u = f$, where $\mathcal{L}$ is a closed operator on a Banach space $\mathcal{F}$ which commutes with operators from a C_0-group of isometric operators $V(t)$. This operator equation

Received March 7, 2011.
1991 *Mathematics Subject Classification.* Primary: 34G10; Secondary: 35K90.
Key words and phrases. Operator equation, C_0-semigroup, isometric group, almost periodic, spectrum, equation spectrum, regular admissibility.

includes, e.g., the following operator equation

$$\mathcal{D}u - \mathcal{B}u = f,$$

where $\mathcal{D}$ is the generator of $V(t)$ and $\mathcal{B}$ is a closed linear operator which commutes with $V(t)$, as well as other classes of equations which occur in applications to functional-differential equations. We introduce the notion of *equation spectrum* Σ associated with this operator equation and prove that if the equation spectrum is countable and f is an almost periodic (asymptotically almost periodic) element (under the group $V(t)$), then the solution u is almost periodic (resp., asymptotically almost periodic) (Theorems 3.20 and 3.21). The discrete version of this result also is presented in Section 4. The results of this paper give a unified approach to questions of almost periodicity of solutions of linear time-independent functional-differential or difference equations in Banach spaces, which we present in Section 5. They have potential for further applications to the theory of asymptotic behaviour of solutions of functional-differential and difference equations and also have, in our opinion, independent interest.

Throughout this paper, we denote by $BC(\mathbb{R}, E)$, $BUC(\mathbb{R}, E)$ and $AP(\mathbb{R}, E)$ the Banach spaces of bounded continuous, bounded unniformly continuous and almost periodic functions on $\mathbb{R}$ with values in a Banach space E, respectively. If A is a linear operator on E, then $D(A)$, $\sigma(A)$ and $\rho(A)$ will denote the domain, spectrum and resolvent of A, respectively.

2. Preliminaries: spectrum of elements under isometric groups

For convenience of the reader, we recall the definition of the spectrum of a continuous bounded function on $\mathbb{R}$ with values in a Banach space E and the well known properties of the spectrum, that will be used throughout the paper [1]. For details, the reader is referred to the books [4, 8, 13, 19], or papers [21, 25].

The *Beurling spectrum*, $Sp_B(f)$, of f is defined as the hull of the ideal I_f of the Banach algebra $L^1(\mathbb{R})$, where

$$I_f := \{g \in L^1(\mathbb{R}) : g * f = 0\}.$$

Since maximal ideals of $L^1(\mathbb{R})$ are identified with points $\lambda \in \mathbb{R}$, we have

$$Sp_B(f) := \{\lambda \in \mathbb{R} : \hat{g}(\lambda) = 0 \text{ for all } g \in I_f\}.$$

We will frequently use the following well known equivalent characterization of the Beurling spectrum.

Proposition 2.1. *A point $\lambda \in \mathbb{R}$ is in $Sp_B(f)$ if and only if for every neighborhood $\mathcal{U}$ of λ there exists a function $\phi \in L^1(\mathbb{R})$ such that $supp\, \phi \subset \mathcal{U}$ and $\phi * f \neq 0$.*

Beside the Beurling definition of the spectrum, there is another definition via the notion of Carleman transform. Define, for each bounded continuous function

[1] The spectrum is defined and all its properties hold for strongly measurable bounded functions

f, its Carleman transform by

$$\tilde{f}(\lambda) = \begin{cases} \int_0^\infty e^{i\lambda t} dt & \text{if } Im\,\lambda > 0 \\ \int_{-\infty}^0 e^{i\lambda t} dt & \text{if } Im\,\lambda < 0. \end{cases}$$

Thus, $\tilde{f}$ is an analytic function in $\{\lambda \in \mathbb{C} : Im\,\lambda \neq 0\}$ [2]. The *Carleman spectrum*, $Sp_C(f)$, is defined as the set of $\lambda \in \mathbb{R}$ such that $\tilde{f}$ does not have an analytic continuation into a neighborhood of λ.

It is well known that $Sp_B(f) = Sp_C(f)$ for any function $f \in BUC(\mathbb{R}, E)$ (see e.g. [4, 19, 25]), so that we can denote the spectrum of f simply by $Sp(f)$. In the next statement we gather the well known properties of $Sp(f)$ (we denote by $L_c(\mathbb{R})$ the set of functions $\phi \in L^1(\mathbb{R})$ Fourier transforms of which have compact support).

Proposition 2.2. (i) *$Sp(f)$ is a closed subset of $\mathbb{R}$ and $Sp(f) = \emptyset$ if and only if $f = 0$;*

(ii) *$Sp(f_t) = Sp(f)$;*

(iii) *$Sp(f) = \{\lambda\}$ if and only if $f(t) = e^{i\lambda t}x$ for some $x \in E$;*

(iv) *$Sp(f + g) \subseteq Sp(f) \cup Sp(g)$;*

(v) *$Sp(\phi * f) \subseteq supp\,\phi \cap Sp(f)$, $Sp(f - \phi * f) \subseteq Sp(f) \cap supp\,(1 - \hat{\phi})$, for every $\phi \in L^1(\mathbb{R})$;*

(vi) *$Sp(f) = \overline{\cup_{\phi \in L_c(\mathbb{R})} Sp(\phi * f)}$;*

(vii) *If $\|f_n - f\| \to 0$ as $n \to \infty$ and $Sp(f_n) \subset \Lambda$ for all n, then $Sp(f) \subseteq \overline{\Lambda}$;*

(viii) *If B is a closed operator such that $f(t) \in D(B)$ $\forall t$, and $Bf(t) \in BC(\mathbb{R}, E)$, then $Sp(Bf) \subseteq Sp(f)$. Moreover, if $\mathcal{B} : BC(\mathbb{R}, E) \to BC(\mathbb{R}, E)$ is a closed operator which commutes with the translations, then $Sp(\mathcal{B}f) \subseteq Sp(f)$ for all $f \in BC(\mathbb{R}, E)$.*

Let $AP(\mathbb{R}, E)$ denote the space of Bohr almost periodic functions on $\mathbb{R}$ with values in E. It is well known that $AP(\mathbb{R}, E)$ is a closed, translation-invariant subspace of $BUC(\mathbb{R}, E)$. The following notion of *almost periodic spectrum* of a function f is due to Loomis [15].

Definition 2.3. The almost periodic spectrum (or *ap*-spectrum) is defined by

$$Sp_{ap}(f) := \{\lambda \in \mathbb{R} : \text{ for every neighborhood } \mathcal{U} \text{ of } \lambda$$
$$\text{there exists } \phi \in L_c(\mathbb{R}) \text{ such that } supp\,\hat{\phi} \subset \mathcal{U} \text{ and } \phi * f \notin AP(\mathbb{R}, E)\}.$$

Observe that many properties of the spectrum in Proposition 2.2 remain valid, with corresponding modifications, for the *ap*-spectrum.

Proposition 2.4. (i) *$Sp_{ap}(f)$ is a closed subset of $\mathbb{R}$ and $Sp_{ap}(f) = \emptyset$ if and only if $f \in AP(\mathbb{R}, E)$;*

(ii) *$Sp_{ap}(f_t) = Sp_{ap}(f)$;*

(iii) *$Sp_{ap}(f + g) \subseteq Sp_{ap}(f) \cup Sp_{ap}(g)$;*

(iv) *$Sp_{ap}(\phi * f) \subseteq supp\,\phi \cap Sp_{ap}(f)$, $Sp_{ap}(f - \phi * f) \subseteq Sp_{ap}(f) \cap supp\,(1 - \hat{\phi})$, for every $\phi \in L^1(\mathbb{R})$;*

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[2] More exactly, $\tilde{f}$ represents a pair of analytic functions

(v) $Sp_{ap}(f) = \overline{\cup_{\phi \in L_c(\mathbb{R})} Sp_{ap}(\phi * f)}$;

(vi) *If* $\|f_n - f\|_\infty \to 0$ *as* $n \to \infty$ *and* $Sp_{ap}(f_n) \subset \Lambda$ *for all* n*, then* $Sp_{ap}(f) \subseteq \overline{\Lambda}$;

(vii) *If* B *is a closed operator such that* $f(t) \in D(B) \forall t$*, and* $Bf(t) \in BC(\mathbb{R}, E)$*, then* $Sp_{ap}(Bf) \subseteq Sp_{ap}(f)$*. Moreover, if* $\mathcal{B} : BC(\mathbb{R}, E) \to BC(\mathbb{R}, E)$ *is a closed operator which commutes with the translations, then* $Sp_{ap}(\mathcal{B}f) \subseteq Sp_{ap}(f)$ *for all* $f \in BC(\mathbb{R}, E)$*.*

The following characterization of the *ap*-spectrum will not be used in the sequel but serves to add information on spectra of a function.

Proposition 2.5. *The following identity holds*

$$Sp_{ap}(f) = \cap_{g \in AP(\mathbb{R},E)} Sp(f + g).$$

Proof. First we show that $Sp_{ap} \subseteq Sp(f+g)$ for every $g \in AP(\mathbb{R}, E)$. Assume that $\lambda \notin Sp(f + g)$. Then there exists a neighborhood $\mathcal{U}$ of λ such that if $\phi \in L_c(\mathbb{R})$ and $supp\,\hat{\phi} \subset \mathcal{U}$, then $\phi * (f + g) = 0$. Hence $\phi * f = -\phi * g$ is almost periodic for all ϕ with $supp\,\hat{\phi} \subset \mathcal{U}$, which means $\lambda \notin Sp_{ap}(f)$.

Conversely, assume that $\lambda \notin Sp_{ap}(f)$. Then one can choose neighborhoods $\mathcal{U}$ and $\mathcal{V}$ of λ, with $\overline{\mathcal{U}} \subset \mathcal{V}$ and $\phi \in L_c(\mathbb{R})$ such that $supp\,\hat{\phi} \subset \mathcal{V}$, $\hat{\phi}|\overline{\mathcal{U}} = 1$ and $\phi * f \in AP(\mathbb{R}, E)$. Then, for every ψ such that $\hat{\psi} \subset \mathcal{U}$, we have $Sp(\psi * \phi * f - \psi * f) \subset supp\,\hat{\psi} \cap Sp(f - \phi * f) = \emptyset$, since $Sp(f - \phi * f) \cap \mathcal{U} = \emptyset$, by Proposition 2.2, (vii). Thus, $\psi * \phi * f = \phi * f$, so that $\lambda \notin Sp(\phi * f - f)$. Therefore, $\lambda \notin \cap_{g \in AP(\mathbb{R})} Sp(f + g)$. $\square$

Recall that a function $f \in BUC(\mathbb{R}, E)$ is said to be a C_0^+-function, denoted by $f \in C_0^+(\mathbb{R}, E)$, if $\lim_{t\to\infty} \|f(t)\| = 0$. A function f is called *asymptotically almost periodic*, denoted by $f \in AAP(\mathbb{R}, E)$, if there exist an almost periodic function $g \in AP(\mathbb{R}, E)$ and a C_0^+- function $h \in C_0^+(\mathbb{R}, E)$ such that $f(t) = g(t) + h(t)$. Thus, $AAP(\mathbb{R}, E) = AP(\mathbb{R}, E) \oplus C_0^+(\mathbb{R}, E)$. If in Definition 2.3 we replace $AP(\mathbb{R}, E)$ by $AAP(\mathbb{R}, E)$ ($C_0^+(\mathbb{R}, E)$), then we arrive at the notion of *asymptotically almost periodic spectrum*, or *aap*-spectrum, denoted by $Sp_{aap}(f)$, (resp., *asymptotic spectrum,* or C_0^+-spectrum, denoted by $Sp_0(f)$) which have been considered in [21]. For a generalization of this approach see [1,6]. Note that

$$Sp_{aap}(f) \subseteq Sp_{ap}(f) \subseteq Sp(f), \; Sp_{aap}(f) \subseteq Sp_0(f) \subseteq Sp(f),$$

and all the above inclusions are, in general, strict [21].

From Proposition 2.2 (iii) it is easily seen that if $Sp(f)$ is finite, $S(f) = \{e^{i\lambda_k t}, k = 1, 2, ..., m\}$, then f is a trigonometric polynomial, i.e. there exist $x_1, ...x_m$ in E such that $f(x) = \sum_{k=1}^m e^{i\lambda_k t} x_k$. The following theorem is a generalization of this fact to the case when $Sp(f)$ is a discrete set (see [2, 7]).

Theorem 2.6. *If* $Sp(f)$ *is a discrete set, then* f *is almost periodic.*

Now we state the Loomis theorem mentioned in Introduction.

Theorem 2.7. *Let* $f \in BUC(\mathbb{R}, E)$ *and suppose that* $Sp_{ap}(f)$ *is countable. Then* f *is almost periodic if one of the following conditions holds:*

(i) *E does not contain a subspace which is isomorphic to the space c_0 of sequences convergent to zero (or briefly, $E \not\supset c_0$);*

(ii) *$f(\mathbb{R})$ is relatively weakly compact in E;*

(iii) *For every $\lambda \notin Sp(f)$, the function $e^{-i\lambda t}f(t)$ has uniformly convergent means, i.e. the limit*

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T+h}^{T+h} e^{-i\lambda t}f(t)dt$$

converges uniformly in h.

Parts (i)-(ii) of Theorem 2.7 (the almost periodic case) are contained in [14], part (iii) is in [21].

A function f satisfying condition (iii) in Theorem 2.7 is called *totally ergodic.* This terminology is justified by the fact that f satisfies (iii) if and only if the restriction $T_f(t)$ of the translation group $T(t)$ to the subspace M_f spanned by $f_t, t \in \mathbb{R}$, is totally ergodic, i.e. $e^{-i\lambda t}T_f(t)$ is ergodic for every $\lambda \in \mathbb{R}$.

The following is a version of the Loomis theorem for asymptotically almost periodic functions (see [21]).

Theorem 2.8. *Let $f \in BUC(\mathbb{R}, E)$ and suppose that $Sp_{aap}(f)$ is countable. Then f is asymptotically almost periodic if and only if for every $\lambda \notin Sp(f)$, the function $e^{-i\lambda t}f(t)$ has uniformly convergent means. In addition, if the means are equal to 0, then $f \in C_0^+(\mathbb{R}, E)$.*

Now let $\mathcal{F}$ be another Banach space and $V(t), t \in \mathbb{R}$, be a strongly continuous group of isometric operators on $\mathcal{F}$, with generator $\mathcal{D}$. For each element $x \in \mathcal{F}$, put $\mathbf{x}(t) := V(t)x, t \in \mathbb{R}$. Then $\mathbf{x} \in BUC(\mathbb{R}, \mathcal{F})$.

Definition 2.9. (i) An element $x \in \mathcal{F}$ is called almost periodic element with respect to $V(t)$, if the corresponding function $\mathbf{x}(t)$ is almost periodic;

(ii) The spectrum (almost periodic spectrum) of and element x in $\mathcal{F}$, under the isometric group $V(t)$, is defined by

$$Sp^V(x) = Sp(\mathbf{x}) \text{ (resp. } Sp^V_{ap}(x) = Sp_{ap}(\mathbf{x})).$$

It is well known that $iSp^V(x) = \sigma(\mathcal{D}|M_x)$, where M_x is the closure of the span of $V(t)x$ (see [25]).

Let Λ be a closed subset of $\mathbb{R}$ and $M(\Lambda) = \{x \in \mathcal{F} : Sp(x) \subset \Lambda\}$. The following proposition contains well known results on isometric groups. We refer the reader to [5, 16, 17] for details.

Proposition 2.10. (i) *$M(\Lambda)$ is a closed subspace which is an invariant subspace for $V(t), \forall t$, and $\mathcal{D}$;*

(ii) *$\sigma(\mathcal{D}|M(\Lambda)) \subseteq \Lambda$; if Λ is compact, then $\mathcal{D}|M(\Lambda)$ is bounded.*

(iii) *$M(\Lambda)$ is invariant with respect to every closed operator B which commutes with $V(t)$.*

(iv) *$M(\{\lambda\}) = \{x \in \mathcal{F} : Sp(x) = \{\lambda\}\} = \{x : Dx = i\lambda x\}$.*

(v) *The span of $M(\Lambda)$, where Λ runs over all compact subsets, is dense in $\mathcal{F}$.*

It is well known that an element x in $\mathcal{F}$ is an almost periodic element under the group $V(t)$ if and only if the set $\{V(t)x : t \in \mathbb{R}\}$ is relatively compact. The group $V(t)$ is called *almost periodic group* if every x in $\mathcal{F}$ is an almost periodic element under $V(t)$.

More gerenally, assume that $\mathcal{K}$ is a closed subspace of $\mathcal{F}$ which is invariant with respect to $V(t), t \in \mathbb{R}$. Let $\widehat{\mathcal{F}} = \mathcal{F}/\mathcal{K}$ be the corresponding quotient space, that is $\widehat{\mathcal{F}} := \{\hat{x} := x + \mathcal{K} : x \in \mathcal{F}\}$ with the norm defined by

$$\|\hat{x}\| = \inf\{\|x + y\| : y \in \mathcal{K}\}.$$

Define operators $\widehat{V}(t) : \widehat{\mathcal{F}} \to \widehat{\mathcal{F}}$ by $\widehat{V}(t)(\hat{x}) = (\widehat{V(t)x})$. Then, as directly verified, $\widehat{V}(t), t \in \mathbb{R}$ is an isometric C_0-group on $\widehat{\mathcal{F}}$. Define $\mathcal{K}$-spectrum of an element $x \in \mathcal{F}$, with respect to $V(t)$, by

$$Sp^V_{\mathcal{K}}(x) = Sp^{\widehat{V}}(\hat{x}).$$

Proposition 2.11. *The following are equivalent*

(i) $x \in \mathcal{K}$.

(ii) $Sp^V_{\mathcal{K}}(x) = \emptyset$.

Proof. If $x \in \mathcal{K}$, then $\hat{x} = 0$, hence $Sp^V_{\mathcal{K}}(x) = Sp^{\widehat{V}}(\hat{x}) = \emptyset$. Converserly, if $Sp^V_{\mathcal{K}}(x) = Sp^{\widehat{V}}(\hat{x}) = \emptyset$, then $\hat{x} = 0$, hence $x \in \mathcal{K}$. □

In the sequel, when the underlying operator group $V(t)$ is fixed (and, as a rule, it is the translation group in an appropriate function space), we will simply write $Sp(x)$, $Sp_{ap}(x)$, $Sp_{\mathcal{K}}(x)$ instead of $Sp^V(x)$, $Sp^V_{ap}(x)$, $Sp^V_{\mathcal{K}}(x)$, respectively.

For the special case when $\mathcal{F} = BUC(\mathbb{R}, E)$, $V(t)$ is the translation group and $\mathcal{K}$ is one of the following subspaces

(i) $\mathcal{K} = AP(\mathbb{R}, E)$,

(ii) $\mathcal{K} = AAP(\mathbb{R}, E)$,

(iii) $\mathcal{K} = C_0^+(\mathbb{R}, E)$,

the corresponding notion of $\mathcal{K}$-spectrum $Sp_{\mathcal{K}}(f), f \in \mathcal{F}$, coincides with $Sp_{ap}(f)$, $Sp_{aap}(f)$ and $Sp_0(f)$, respectively.

The Loomis theorem when applied to the isometric group $V(t)$ implies that an element x is almost periodic if $Sp_{ap}(x)$ is countable and either $\mathcal{F} \not\supset c_0$ or $e^{-i\lambda t}V(t)x$ has convergent means for each $\lambda \notin Sp(x)$. The latter is equivalent to ergodicity of the group $e^{-\lambda t}V(t)$ restricted to M_x, for each $\lambda \notin Sp(x)$ [3]. In the sequel, elements x satisfying this condition are called *totally ergodic*, and $V(t)$ is called totally ergodic if $V(t)x$ is totally ergodic for each $x \in E$. Thus, if $\sigma(\mathcal{D})$ is countable and either $\mathcal{F} \not\supset c_0$ or $V(t)$ is totally ergodic, then $V(t)$ is an almost periodic group.

3. The equation spectrum and almost periodicity

Let $\mathcal{F}$ be a Banach space, $V(t)$ an isometric C_0-group on $\mathcal{F}$, with the generator $\mathcal{D}$, and $\mathcal{B}$ be a closed, generally unbounded, operator on $\mathcal{F}$, which commutes

[3]It is well known that if $\lambda \notin Sp(x)$, then the group $e^{-i\lambda t}V(t)|M_x$ is always ergodic.

with $V(t), t \in \mathbb{R}$. The difference $\mathcal{D} - \mathcal{B}$ is defined in the usual way, namely $D(\mathcal{D} - \mathcal{B}) = D(\mathcal{D}) \cap D(\mathcal{B})$ and $(\mathcal{D} - \mathcal{B})u = \mathcal{D}u - \mathcal{B}u$ for $u \in D(\mathcal{D} - \mathcal{B})$. This implies that, for every function $\phi \in L^1(\mathbb{R})$, $\mathcal{B}$ commutes with $\hat{\phi}(T)$, where

$$\hat{\phi}(T) = \int_{\mathbb{R}} \phi(t)V(t)dt.$$

In particular, if $x \in D(\mathcal{B})$, then $(\phi * \mathbf{x})(t) \in D(\mathcal{B})$ and $\mathcal{B}(\phi * \mathbf{x})(t) = (\phi * (\mathcal{B}\mathbf{x}))(t)$ (where $\mathcal{B}\mathbf{x}(t) = V(t)\mathcal{B}x$).

Lemma 3.1. *$\mathcal{D} - \mathcal{B}$ is closable.*

Proof. Let $x_n \in D(\mathcal{D}) \cap D(\mathcal{B})$, $x_n \to 0$ and $\mathcal{D}x_n - \mathcal{B}x_n \to y$. Then $\mathcal{D}V(t)x_n - \mathcal{B}V(t)x_n \to V(t)y$ uniformly. Take $\phi \in L^1(\mathbb{R})$ such that $supp\hat{\phi}$ is compact. Then $\mathcal{D}[(\phi * \mathbf{x_n})(t)] - \mathcal{B}[\phi * \mathbf{x_n}(t)]$ converges uniformly to $(\phi * \mathbf{y})(t)$. But $\mathcal{D}[(\phi * \mathbf{x_n})] \to 0$, hence $\mathcal{B}[(\phi * x_n)(t)] \to -(\phi * \mathbf{y})(t)$, so that $(\phi * \mathbf{y})(t) = 0$ $(\forall t)$. This implies $y = 0$. □

Note that, in general, $\mathcal{D} - \mathcal{B}$ is not closed. Let $(\mathcal{D} - \mathcal{B})^\sim$ denote the closure of $(\mathcal{D} - \mathcal{B})$.

We consider the operator equation

$$\mathcal{D}u - \mathcal{B}u = f, \; f \in \mathcal{F} \tag{3.1}$$

Definition 3.2. An element u in $\mathcal{F}$ is called a mild solution of (3.1) if $u \in D((\mathcal{D} - \mathcal{B})^\sim)$ and $(\mathcal{D} - \mathcal{B})^\sim u = f$.

We are interested in the almost periodicity of solutions of (3.1) or, more generally, in whether a solution u of (3.1) belongs to a corresponding subspace $\mathcal{K}$, where $\mathcal{K}$ is a closed subspace of $\mathcal{F}$ which is invariant with respect to $V(t)$ and every closed linear operator which commutes with $V(t), t \in \mathbb{R}$. It is not difficult to see that if the solution u is almost periodic or $u \in \mathcal{K}$, then so is f. Therefore, it is natural to assume that the element f is almost periodic (resp, $f \in \mathcal{K}$). Note that if $\mathcal{D} - \mathcal{B}$ is invertible (i.e. has a bounded inverse), then the solution u is given by $u = (\mathcal{D} - \mathcal{B})^{-1}f$, hence u is almost periodic (resp., $u \in \mathcal{K}$).

Definition 3.3. Let Λ be a closed subset of $\mathbb{R}$. The space $M(\Lambda)$ is called *regularly admissible* if for every $f \in M(\Lambda)$ there exists a unique mild solution u in $M(\Lambda)$ of (3.1).

Example 3.4. Let E be a Banach space, $\mathcal{F} = BUC(\mathbb{R}, E)$, $V(t)$ be the translation group on $\mathcal{F}$, with the generator $\mathcal{D}$, and let A be a closed linear operator on E. Then A generates a closed linear operator $\mathcal{B}$ on $\mathcal{F}$ by

$$D(\mathcal{B}) := \{f \in \mathcal{F} : f(t) \in D(A) \; \forall t \text{ and } Af(t) \in \mathcal{F}\}, \quad (\mathcal{B}f)(t) = Af(t), \; f \in D(\mathcal{B}).$$

It is easily seen that $\mathcal{B}$ is a closed operator which commutes with $V(t)$, and one can show, without difficulty, that a function $u \in \mathcal{F}$ is a mild solution to (3.1) in our sense if and only if u is a mild solution of the differential equation

$$u'(t) = Au(t) + f(t)$$

in the standard sense, i.e. $\int_0^t u(s)ds \in D(A)$ and

$$u(t) = u(0) + A\int_0^t u(s)ds + \int_0^t f(s)ds,\ t \in \mathbb{R}.$$

In particular, if A is the generator of a C_0-semigroup $T(t)$, then $u \in \mathcal{F}$ is a mild solution in our definition if and only if u is a mild solution in the standard sense of the theory of C_0-semigroups, i.e. u satisfies

$$u(t) = T(t-s)u(s) + \int_s^t T(t-\tau)f(\tau)d\tau,\ (t \geq s)$$

(see the proof of Proposition 5.3).

Lemma 3.5. *$M(\Lambda)$ is regularly admissible if and only if $(\mathcal{D}-\mathcal{B})|M(\Lambda)$ is invertible.*

Proof. Let $M(\Lambda)$ be regularly admissible and define an operator $K : M(\Lambda) \to M(\Lambda)$ by $Kf = u$, where u is the unique solution in $M(\Lambda)$ of (3.1). We show that K is closable. Let $f_n \in M(\Lambda)$, $f_n \to 0$ and $(\mathcal{D}-\mathcal{B})^{\sim}u_n = f_n, u_n \to u$. Since $(\mathcal{D}-\mathcal{B})^{\sim}$ is closed, we have $u \in D((\mathcal{D}-\mathcal{B})^{\sim})$ and $(\mathcal{D}-\mathcal{B})^{\sim}u = 0$. But this implies $u = 0$, because of the uniqueness of solutions in $M(\Lambda)$. By the closed graph theorem, K is a bounded linear operator. Therefore $[(\mathcal{D}-\mathcal{B})|M(\Lambda)]^{\sim}Kf = f$ for all $f \in M(\lambda)$. Since $[(\mathcal{D}-\mathcal{B})|M(\Lambda)]^{\sim}$ is injective in $M(\Lambda)$, this implies that $([(\mathcal{D}-\mathcal{B})|M(\Lambda)]^{\sim}$ is invertibe and $([(\mathcal{D}-\mathcal{B})|M(\Lambda)]^{\sim})^{-1} = K$.

Conversely, assume that $[(\mathcal{D}-\mathcal{B})|M(\Lambda)]^{\sim}$ is invertible and let K be its inverse. Then, for every $f \in M(\Lambda)$, it is easily seen that $u = Kf$ satisfies (3.1). □

Lemma 3.6. *Assume $M(\Lambda)$ is regularly admissible.*

(i) *If $f \in M(\Lambda)$ and u is the corresponding mild solution of* (3.1) *in $M(\Lambda)$, then $Sp(u) \subseteq Sp(f)$;*

(ii) *If Λ_0 is a closed subset of Λ, then $M(\Lambda_0)$ is also regularly admissible;*

(iii) *If f is an almost periodic element (resp., $f \in \mathcal{K}$), then the corresponding solution $u \in M(\Lambda)$ also is almost periodic (resp., $u \in \mathcal{K}$).*

Proof. (i) Assume that $\lambda \notin Sp(f)$. Then there exists a neighborhood $\mathcal{U}$ of λ such that if $supp\,\hat{\phi} \subset \mathcal{U}$, then $\phi * \mathbf{f} = 0$. Since $(\phi * \mathbf{u})(t)$ is the mild solution in $M(\Lambda)$ of the homogeneous equation (3.1) (with f replaced by 0), we have $(\phi * \mathbf{u})(t) = 0 \forall t$. Therefore, $\lambda \notin Sp(u)$.

(ii) follows directly from (i).

(iii) By Lemma 3.5, $[(\mathcal{D}-\mathcal{B})|M(\Lambda)]^{\sim}$ is invertible. Let K be its inverse. It is easily seen that K commutes with $V(t)|M(\Lambda)$. Hence, if $V(t)f$ is relatively compact, then so is $V(t)Kf = V(t)u$. If $f \in \mathcal{K}$, then $u = Kf \in \mathcal{K}$. □

The following lemma is contained in [3, Theorem 7.2] (cf. also [26, Lemma 22]).

Lemma 3.7. *Let A be a bounded linear operator and B be a closed operator which commutes with A. If $\sigma(A) \cap \sigma(B) = \emptyset$, then $A - B$ is invertible.*

Lemma 3.5 and Lemma 3.7 imply the following result.

Lemma 3.8. *Assume that* Λ *is a compact subset of* $\mathbb{R}$. *If* $\sigma(\mathcal{B}|M(\Lambda))\cap\sigma(\mathcal{D})\cap i\Lambda = \emptyset$, *then* $M(\Lambda)$ *is regularly admissible.*

Lemma 3.9. $\sigma(\mathcal{B}|M(\Lambda)) \subseteq \sigma(\mathcal{B})$.

Proof. It is well known that if a subspace M of the Banach space $\mathcal{F}$ is an invariant subspace with respect to $\mathcal{B}$ and to all resolvents $(\lambda - \mathcal{B})^{-1}, \lambda \in \rho(\mathcal{B})$, then $\sigma(\mathcal{B}|M) \subset \sigma(\mathcal{B})$. Since $M(\Lambda)$ is invariant with respect to $(\lambda - \mathcal{B})^{-1}$, the lemma follows. □

From Lemma 3.9 and Lemma 3.8 we have the following corollary.

Corollary 3.10. *Assume that* Λ *is a compact subset of* $\mathbb{R}$. *If* $\sigma(\mathcal{B})\cap\sigma(\mathcal{D})\cap i\Lambda = \emptyset$, *then* $M(\Lambda)$ *is regularly admissible.*

In the case when $\mathcal{D} = \frac{d}{dt}$ and $\mathcal{B}$ is a functional-differential operator (in $\mathcal{F} := BUC(\mathbb{R}, E)$), the standard notion of equation spectrum is obtained by applying the Fourier transform to both parts of the equation. Below we introduce the notion of equation spectrum, which genegalizes this notion of equation spectrum for functional differential equations to general operator equations of the form (3.1).

Definition 3.11. The equation spectrum of (3.1) is

$$\begin{aligned}\Sigma(\mathcal{D}, \mathcal{B}) := \{\lambda \in \mathbb{R} : &\text{ for every neighborhood } \mathcal{U} \text{ of } \lambda,\\ &(\mathcal{D} - \mathcal{B})|M(\overline{\mathcal{U}}) \text{ is not invertible}\}\end{aligned}$$

Since the operators $\mathcal{D}, \mathcal{B}$ are fixed, we will denote the equation spectrum simply by Σ. The following lemma follows immediately from Corollary 3.10.

Lemma 3.12. $i\Sigma \subset [\sigma(\mathcal{B}) \cap \sigma(\mathcal{D})]$.

As we will see in examples 3.13-3.17 below, $\sigma(\mathcal{B})$ is, in general, too large for the purpose of establishing effective spectral criteria of almost periodicity.

Example 3.13. Let E be a Banach space, $\mathcal{F} := BUC(\mathbb{R}, E)$ be the space of bounded uniformly continuous functions on $\mathbb{R}$ with values in E, $V(t)$ be the translation group in $BUC(\mathbb{R}, E)$, $\mathcal{D}$ be the generator of $V(t)$ (the differentiation operator) and A be a closed linear operator on E. The operator A generates a closed operator $\mathcal{B}$ on $BUC(\mathbb{R}, E)$ by $(\mathcal{B}f)(t) = Af(t)$. Then the equation spectrum Σ of equation $\mathcal{D}u - \mathcal{B}u = f$ is

$$\Sigma = \{\lambda \in \mathbb{R} : i\lambda \in \sigma(A)\},$$

so that $i\Sigma = \sigma(A) \cap i\mathbb{R}$. This follows from [24], Theorem 3.3.

Example 3.14. Let $\mathcal{F} = BUC(\mathbb{R}, \mathbb{C})$, $\mathcal{D}$ be the differentiation operator and $\mathcal{B} : \mathcal{F} \to \mathcal{F}$ be defined by $(\mathcal{B}f) = -i\pi f(t-1)$ (i.e. $\mathcal{B} = QV(-1)$ where Q is multiplication by $-i\pi$ and $V(-1)$ is right translation by the unit). The operator equation $\mathcal{D}u - \mathcal{B}u = f$ is the same as the delay equation

$$u'(t) = -i\pi u(t-1) + f(t).$$

It is not difficult to calculate that the equation spectrum is $\Sigma = \{-\pi\}$, while $\sigma(\mathcal{B}) \cap \sigma(\mathcal{D} \cap i\mathbb{R} = \{-i\pi, i\pi\}$, so that $i\Sigma \neq \sigma(B) \cap \sigma(\mathcal{D})$.

Example 3.15. Let H be a finite or infinite dimensional Hilbert space, with an orthonormal basis $\{e_k\}_{k=1}^N, N \in \mathbb{N} \cup \{\infty\}$, $\mathcal{D}$ and $\mathcal{B}$ be diagonal operators defined by $\mathcal{D}e_k = id_k e_k,\ \mathcal{B}e_k = ib_k e_k, 1 \leq k \leq N$. Then one can easily show that the equation spectrum for (3.1) consists of those b_k's which are equal to the corresponding d_k's.

Example 3.16. Let $\mathbb{B}$ be the space of Bohr almost periodic functions on $\mathbb{R}$, equipped with the scalar product

$$\langle f, g \rangle := \lim_{T\to\infty} \int_{-T}^{T} f(t)\overline{g(t)}dt,$$

and let $H = AP(\mathbb{R})$ be the completion of $\mathbb{B}$. Then H is a nonseparable Hilbert space with the orthonormal basis $e_\lambda := e^{i\lambda t},\ \lambda \in \mathbb{R}$. Define $\mathcal{D}$ and B on H by:

$$\mathcal{D}e_\lambda = i\lambda e_\lambda,\ Be_\lambda = b(\lambda)e_\lambda,\ \lambda \in \mathbb{R},$$

where $b(\lambda)$ is an arbitrary continuous function on $\mathbb{R}$. Then B extends to a closed operator on H with domain

$$D(B) = \left\{ f = \sum_\lambda a(\lambda, f)e_\lambda \in H :\ \sum_\lambda |b(\lambda)a(\lambda, f)|^2 < \infty \right\}, \tag{3.2}$$

(where the summation in (3.2) is over a countable set of λ's), and

$$Bf = \sum_\lambda [b(\lambda)a(\lambda, f)]e_\lambda.$$

One can show, using a standard argument of the theory of operators (in particular, the Parseval's equality) that the equation spectrum Σ for the equation $\mathcal{D}u - \mathcal{B}u = f$ is given by

$$\Sigma = \{\lambda : b(\lambda) = i\lambda\}.$$

Therefore, $i\Sigma$ coincides with the set of those $b(\lambda)$ that $b(\lambda) = i\lambda$, while the spectrum $\sigma(\mathcal{B})$ of $\mathcal{B}$ is the closure of the whole range of $b(\lambda)$ (and $\sigma(\mathcal{D}) = i\mathbb{R}$).

Example 3.17. Consider the Volterra equation

$$u'(t) = Au(t) + \int_0^\infty dB(\tau)u(t-\tau) + f(t), \tag{3.3}$$

where A is a closed linear operator on a Banach space E, with dense domain $D(A)$, $\{B(t)\}_{t\geq 0}$ is a family of closed linear operators in E with $D(B(t)) \supset D(A)$ for all $t \geq 0$ such that $B \in BV(\mathbb{R}_+, \mathbb{B}(Y, E))$ (the space of $\mathbb{B}(Y, E)$-valued functions of bounded variation over $\mathbb{R}_+$) and $Y = D(A)$ with the graph norm. (3.3) is associated with an operator equation $\mathcal{D}u - \mathcal{B}u = f$, where $\mathcal{D}f = f'$ and $\mathcal{B}$ is defined by

$$(\mathcal{B}u)(t) = Au(t) + \int_0^\infty dB(\tau)u(t-\tau),\ u \in BUC(\mathbb{R}, E).$$

One can show that, under well-posedness conditions, $\mathcal{B}$ is closable and its closure, which is denoted by the same symbol $\mathcal{B}$, commutes with the translation group

$V(t)$ (hence with $\mathcal{D}$). It follows from [20, Proposition 1 and Theorem 1], that the equation spectrum Σ of the equation $(\mathcal{D} - \mathcal{B})u = f$ is

$$\Sigma = \{\lambda \in \mathbb{R} : [i\lambda - A - \widehat{dB}(i\lambda)] \text{ is not invertible}\}.$$

On the other hand, $\sigma(\mathcal{D}) = i\mathbb{R}$, and $\sigma(\mathcal{B})$ contains the complement of the set

$$\Omega := \{\lambda \in \mathbb{R} : [i\lambda - A - \widehat{dB}(i\xi)] \text{ are invertible for all } \xi \in \mathbb{R} \text{ and } \sup_{\xi\in\mathbb{R}} \|[i\lambda - A - \widehat{dB}(i\xi)]^{-1}\| < \infty\}.$$

Examples 3.16 and 3.17 demonstrate that, in general, the equation spectrum Σ can be significantly smaller than $\sigma(B)$.

Lemma 3.18. *Let $f \in \mathcal{F}$ and u be a mild solution of* (3.1). *Then*

(i) *$Sp(u) \subseteq \Sigma \cup Sp(f)$; in particular, if $f = 0$ (i.e. $\mathcal{L}u = 0$), then $Sp(u) \subseteq \Sigma$;*

(ii) *$Sp_{ap}(u) \subseteq \Sigma \cup Sp_{ap}(f)$; in particular, if f is almost periodic, then $Sp_{ap}(u) \subseteq \Sigma$.*

(iii) *$Sp_{\mathcal{K}}(u) \subseteq \Sigma \cup Sp_{\mathcal{K}}(f)$; in particular, if $f \in \mathcal{K}$, then $Sp_{\mathcal{K}}(u) \subseteq \Sigma$.*

Proof. (i) Let $\lambda \in \mathbb{R}$ and $\lambda \notin (\Sigma \cup Sp(f)$. Then there exists a neighborhood $\mathcal{U}$ of λ such that $\overline{\mathcal{U}} \cap \Sigma = \emptyset$, $M(\overline{\mathcal{U}})$ is regularly admissible and $\phi * \mathbf{f} = 0$ whenever $supp\,\hat{\phi} \subset \mathcal{U}$. If u is a solution of (3.1), then $(\mathcal{D} - \mathcal{B})(\phi * \mathbf{u})(t) = \phi * \mathbf{f}(t) = 0$. This implies $\phi * \mathbf{u} = 0$, since $M(\overline{\mathcal{U}})$ is regularly admissible. Hence $\lambda \notin Sp(u)$.

The proofs of (ii)-(iii) are analogous. $\square$

From Theorems 2.6, 2.7, 2.8 and Lemma 3.18 we immediately obtain the following main results of this paper.

Theorem 3.19. *Assume that $\Sigma \cup Sp(f)$ is a discrete set and u is a mild solution of* (3.1). *Then u is almost periodic.*

Theorem 3.20. *Assume that Σ is countable, f is an almost periodic element and u is a mild solution of* (3.1). *Then $Sp_{ap}(u)$ is countable. In particular, u is almost periodic, provided one of the following conditions holds:*

(i) *$\mathcal{F}$ does not contain a copy of c_0;*

(ii) *$\{V(t)u : t\mathbb{R}\}$ is weakly relatively compact;*

(iii) *$V(t)x$ is totally ergodic.*

Theorem 3.21. *Assume that Σ is countable, $f \in \mathcal{K}$ and u is a mild solution of* (3.1). *Then $Sp_{\mathcal{K}}(u) \subset \Sigma$, so that it is countable.*

Remark 3.22. Wishing to emphasize the connection between our results and previously known results on almost periodicity of differential equation $u'(t) = Au(t)+f(t)$, we have chosen to consider operator equations of the form $\mathcal{D}u-\mathcal{B}u = f$. However, all results in this section have a direct generalization to operator equation of the form

$$\mathcal{L}u = f, \tag{3.4}$$

where $\mathcal{L}$ is a closed linear operator on $\mathcal{F}$ which commutes with $V(t)$. A subspace $M(\Lambda)$ is said to be regularly admissible if for every $f \in M(\Lambda)$, (3.4) has a unique solution in $M(\Lambda)$, or, equivalently, if $\mathcal{L}|M(\Lambda)$ is invertible. The equation spectrum of (3.4) is the set of real numbers λ such that for every neighborhood $\mathcal{U}$ of λ, $M(\overline{\mathcal{U}})$ is not regularly admissible. If u is a solution of (3.4), then $Sp(u) \subseteq \Sigma \cup Sp(f)$ and analogous inclusions hold for *ap*-, *aap*- and C_0^+-spectra (generalization of Lemma 3.18). Therefore, all theorems in this section remain true for (3.4).

4. The discrete case

The results presented in the previous section have discrete analogs, which are of independent interest. We first recall the notion of spectrum of a bounded sequence in E.

4.1. **Spectrum of sequences.** Let $l^\infty(\mathbb{Z}, E)$ be the Banach space consisting of bounded two-sided sequences $\mathbf{x} = \{x_n\}_{n=-\infty}^{\infty}$, where x_n are elements of a Banach space E. As usual, we denote by $l^1(\mathbb{Z})$ the space of absolutely convergent numerical sequences $\phi = \{\phi_n\}$, with the norm $\|\phi\| = \sum_{n=-\infty}^{\infty} |\phi_n|$. It is well known that $l^1(\mathbb{Z})$ is a commutative regular Banach algebra, with convolution as multiplication. The Gelfand space of $l^1(\mathbb{Z})$ is identified with the unit circle $\Gamma = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, and the Gelfand transform of elements ϕ in $l^1(\mathbb{Z})$ is given by

$$\hat{\phi}(\lambda) = \sum_{n=-\infty}^{\infty} \lambda^{-n} \phi_n.$$

For every $\mathbf{x}$ in $l^\infty(\mathbb{Z}, E)$, define

$$I_{\mathbf{x}} := \{\phi \in l^1(\mathbb{Z}) : \phi * \mathbf{x} = 0\},$$

and define the Beurling spectrum of $\mathbf{x}$, $Sp_B(\mathbf{x})$, as the hull of $I_{\mathbf{x}}$. In other words, $Sp_B(\mathbf{x})$ consists of common zeros of $\hat{\phi}(\lambda)$ for $\phi \in I_{\mathbf{x}}$. The following is another equivalent definition of the Beurling spectrum.

$$\begin{aligned} Sp_B(\mathbf{x}) = \{\lambda \in \Gamma : {} & \text{for every neighborhood } \mathcal{U} \text{ of } \lambda \text{ there exists } \phi \in l^1(\mathbb{Z}) \\ & \text{such that } \mathit{supp}\,\hat{\phi} \subset \mathcal{U},\ \phi * \mathbf{x} \neq 0\}. \end{aligned}$$

As for functions on $\mathbb{R}$, one can also define Carleman transform and Carleman spectrum. For a sequence $\mathbf{x} \in l^\infty(\mathbb{Z}, E)$, the Carleman transform $\tilde{\mathbf{x}}(\lambda)$ is defined by

$$\tilde{\mathbf{x}}(\xi) = \begin{cases} \sum_{n=1}^{\infty} \xi^{n-1} x_n & \text{if } |\xi| < 1, \\ \sum_{n=-\infty}^{0} \xi^{n-1} x_n & \text{if } |\xi| > 1 \end{cases}$$

and the Carleman spectrum $Sp_C(\mathbf{x})$ is the set of all points λ on Γ such that $\tilde{\mathbf{x}}$ does not have an analytic continuation across λ. It is well known that $Sp_C(\mathbf{x}) = Sp_B(\mathbf{x})$ for all $\mathbf{x} \in l^\infty(\mathbb{Z}, E)$, so that we will denote the spectrum of $\mathbf{x}$ simply by $Sp(\mathbf{x})$.

The notions of almost periodic functions and *ap*-spectrum also have analogs for sequences. Recall that a sequence $\mathbf{x} = \{x_n\}$ is called *almost periodic sequence* if the family of shifts $\mathbf{x}_k = \{x_{n+k}\}$ is relatively compact in $l^\infty(\mathbb{Z}, E)$. The set of almost periodic sequences in E is denoted by $AP(\mathbb{Z}, E)$ and is a Banach subspace of $l^\infty(\mathbb{Z}, E)$. A sequence $\mathbf{x} = \{x_n\}$ is called *asymptotically almost periodic* if the family of shifts $\mathbf{x}_k = \{x_{n+k}\}$ is asymptotically relatively compact in $l^\infty(\mathbb{Z}, E)$, and $\mathbf{x}$ is called a C_0^+-sequence if $\lim_{n\to\infty}\|x_n\| = 0$. We denote by $AAP(\mathbb{Z}, E)$ the space of asymptotically almost periodic sequences and by $C_0^+(\mathbb{Z}, E)$ the space of all C_0^+-sequences. It is well known that $AAP(\mathbb{Z}, E) = AP(\mathbb{Z}, E) \oplus C_0^+(\mathbb{Z}, E)$.

A point $\lambda \in \Gamma$ is said to be in the *ap*-spectrum $Sp_{ap}(\mathbf{x})$ of a sequence $\mathbf{x}$ if for every neighborhood $\mathcal{U}$ of λ, there exists $\phi \in l^1(\mathbb{Z})$ such that $supp\,\hat{\phi} \subset \mathcal{U}$ and $\phi * \mathbf{x} \notin AP(\mathbb{Z}, E)$. The *aap*-spectrum and C_0^+-spectrum are defined analogously and are denoted by $Sp_{aap}(\mathbf{x})$ (resp., $Sp_0(\mathbf{x})$).

The spectra of sequences have properties analogous to properties of the spectra of functions, i.e. analogs of Propositions 2.2 and 2.4 hold for $Sp(\mathbf{x})$, $Sp_{ap}(\mathbf{x})$, $Sp_{aap}(\mathbf{x})$ and $Sp_0(\mathbf{x})$. We omit the details.

The Loomis theorem states that if $Sp_{ap}(\mathbf{x})$ is countable, then $\mathbf{x} \in AP(\mathbb{Z}, E)$, provided one of the following holds: (i) $X \not\supset c_0$; (ii) $\mathbf{x}$ has weakly relatively compact range; (iii) $\mathbf{x}$ is totally ergodic, i.e.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=h}^{n+h} \lambda^{-k} x_k$$

converges uniformly in $h \in \mathbb{Z}$, for all $\lambda \in Sp(\mathbf{x})$. The analog of Theorem 2.8 states that if $Sp_{aap}(\mathbf{x})$ is countable, then $\mathbf{x} \in AAP(\mathbb{Z}, E)$ if (and only if) $\mathbf{x}$ is totally ergodic.

4.2. **Discrete isometric groups.** Now suppose that $\mathcal{F}$ is another Banach space and V is a single invertible isometry on $\mathcal{F}$. Every element x in $\mathcal{F}$ is associated with a bounded sequence $\mathbf{x} = \{V^n x : n \in \mathbb{Z}\}$. An element x is called *almost periodic element* under V (i.e., under the discrete group $\{V^n : n \in \mathbb{Z}\}$), if the corresponding sequence $\mathbf{x}$ is almost periodic. The spectrum and *ap*-spectrum of x are defined by

$$Sp(x) := Sp(\mathbf{x}),\ Sp_{ap}(x) := Sp_{ap}(\mathbf{x}).$$

Asymptotically almost periodic and C_0^+ elements are defined analogously, as well as the *aap*-spetrum and C_0^+-spectrum. All facts of the theory of continuous groups presented in Section 2 remain true for discrete isometric groups. In particular, for every closed subset Λ of the unit circle Γ, the subspace $M(\Lambda)$ consisting of x such that $Sp(x) \subseteq \Lambda$ is invariant for V and for any closed operator which commutes with V, and $Sp(V|M(\Lambda)) \subseteq \Lambda$. If $Sp_{ap}(x)$ is countable and either $\mathcal{F} \not\supset c_0$ or $\lambda^{-n}V^n x$ has uniformly convergent mean for every $\lambda \in \Gamma \setminus Sp(x)$ (i.e. is totally ergodic), then x is almost periodic (under the discrete group V^n).

4.3. **Almost periodicity of solutions of operator equation.** Let L be a closed, generally unbounded, linear operator on $\mathcal{F}$ which commutes with V and consider the operator equation

$$Lu = f, \tag{4.1}$$

where f, u are in $\mathcal{F}$. A subspace $M(\Lambda)$ is said to be regularly admissible for (4.1) if for every $f \in M(\Lambda)$, (4.1) has a unique solution in $M(\Lambda)$. Equivalently, $M(\Lambda)$ is regularly admissible if $L|M(\Lambda)$ is invertible. Define the equation spectrum of

(4.1) as the set Σ of points $\lambda \in \Gamma$ such that for every neighborhood $\mathcal{U}$ of λ, $M(\overline{\mathcal{U}})$ is not regularly admissible.

Lemma 4.1. *If u is a solution of (4.1), then*
(i) *$Sp(u) \subseteq \Sigma \cup Sp(f)$; in particular, if $Lu = 0$ then $Sp(u) \subseteq \Sigma$;*
(ii) *$Sp_{ap}(u) \subseteq \Sigma \cup Sp_{ap}(f)$; in particular, if f is almost periodic, then $Sp_{ap}(u) \subseteq \Sigma$;*
(iii) *$Sp_{aap}(u) \subseteq \Sigma \cup Sp_{aap}(f)$; in particular, if f is asymptotically almost periodic, then $Sp_{aap}(u) \subseteq \Sigma$;*
(iv) *$Sp_0(u) \subseteq \Sigma \cup Sp_0(f)$; in particular, if f is a C_0^+-element, then $Sp_0(u) \subseteq \Sigma$;*

Theorem 4.2. *Assume that $\Sigma \cup Sp(f)$ is a discrete set and u is a solution of (4.1). Then u is almost periodic.*

Theorem 4.3. *Assume that Σ is countable, f is almost periodic and u is a solution of (4.1). Then $Sp_{ap}(u)$ is countable. In particular, u is almost periodic, provided one of the following holds*
(i) *$\mathcal{F} \not\supset c_0$;*
(ii) *$\{V^n u : n \in \mathbb{Z}\}$ is relatively weakly compact;*
(ii) *u is totally ergodic.*

Theorem 4.4. *Assume that Σ is countable, f is asymptotically almost periodic and u is a solution of (4.1). Then $Sp_{aap}(u)$ is countable. In particular, u is asymptotically almost periodic if (and only if) u is totally ergodic.*

5. Examples

5.1. **First order differential equations.** Consider the differential equation

$$u'(t) = Au(t) + f(t), \tag{5.1}$$

where A is a closed linear operator on a Banach space E. Assume that f is almost periodic (asymptotically almost periodic), and we are interested in conditions under which a solution $u \in BUC(\mathbb{R}, E)$ is almost periodic (resp., asymptotically almost periodic). We will apply results of Section 2 to the case $\mathcal{F} = BUC(\mathbb{R}, E)$, and $(V(t)f)(s) = f(s+t)$ is the translation group on $BUC(\mathbb{R}, E)$, hence $\mathcal{D}$, the generator of $(V(t)$, is the differentiation operator on $BUC(\mathbb{R}, E)$. Let $\mathcal{B}$ be an operator on $BUC(\mathbb{R}, E)$ defined by

$$D(\mathcal{B}) := \{u \in BUC(\mathbb{R}, E) : u(t) \in D(A)\ \forall t \in \mathbb{R} \text{ and } Af(t) \in BUC(\mathbb{R}, E)\},$$

$$(\mathcal{B}f)(t) = Af(t),\ \forall f \in D(\mathcal{B}).$$

As noted in Example 3.13, (5.1) has the form

$$\mathcal{D}u - \mathcal{B}u = f, \tag{5.2}$$

and the equation spectrum Σ of (5.1)-(5.2) coincides with $[-i\sigma(A)] \cap \mathbb{R}$. Therefore, we have the following theorems (cf. [1, 14, 21]).

Theorem 5.1. *Assume that $\sigma(A)\cap i\mathbb{R}$ is countable, f is almost periodic and u is a uniformly continuous bounded mild solution of (5.1). Then $Sp_{ap}(u)$ is countable. In particular, u is almost periodic provided one of the following conditions holds:*
(i) $E \not\supset c_0$;
(ii) *u has relatively compact range;*
(iii) *$u(t)$ is totally ergodic.*

It should be noted that the space $\mathcal{F} = BUC(\mathbb{R}, E)$ always contains c_0, hence Theorem 3.20-(i) does not apply to this case. However, in Theorem 5.1-(i) the assumption $E \not\supset c_0$ is on the Banach space E, not on $\mathcal{F}$. The corresponding conclusion follows simply from the fact that $Sp_{ap}(u)$ is countable and from Theorem 2.7-(i).

Theorem 5.2. *Assume that $\sigma(A) \cap i\mathbb{R}$ is countable, f is asymptotically almost periodic, and u is a uniformly continuous bounded mild solution of (5.1). Then u is asymptotically almost periodic if (and only if) $u(t)$ is totally ergodic.*

5.2. **Second order differential equations.** Consider the differential equation

$$u''(t) = Au(t) + f(t), \tag{5.3}$$

where A is a closed linear operator on a Banach space E. Assume that f is (asymptotically) almost periodic, and we are interested in conditions under which a solution $u \in BUC(\mathbb{R}, E)$ is also (asymptotically) almost periodic. Let $\mathcal{D}$ be as in Example 3.13, $\mathcal{B}$ be an operator on $BUC(\mathbb{R}, E)$ defined by

$$D(\mathcal{B}) := \{u \in BUC(\mathbb{R}, E) : u(t) \in D(A)\ \forall t \in \mathbb{R} \text{ and } Af(t) \in BUC(\mathbb{R}, E)\},$$
$$(\mathcal{B}f)(t) = Af(t),\ \forall f \in D(\mathcal{B}),$$

and $\mathcal{L}_0 = \mathcal{D}^2 - \mathcal{B}$. Exactly as in the proof of Lemma 3.1, we obtain that the operator $\mathcal{L}_0$ is closable. Let $\mathcal{L}$ be the closure of $\mathcal{L}_0$. A function u is called a mild solution of ((5.3)) if $u \in D(\mathcal{L})$ and $\mathcal{L}u = f$. The following proposition shows that a function u is a mild solution in our definition if and only if it is a mild solution in the standard definition (cf. [1]).

Proposition 5.3. *A function u is a mild solution of* (5.3) *if and only if $\int_0^t (t-s)u(s)ds \in D(A)$ and there exist $x, y \in E$ such that*

$$u(t) = x + ty + A\int_0^t (t-s)u(s)ds + \int_0^t (t-s)f(s)ds,\ t \in \mathbb{R}. \tag{5.4}$$

Proof. Assume u is a mild solution of (5.3), i.e. $u \in D(\mathcal{L})$ and $\mathcal{L}u = f$. Then there exist functions $u_n \in D(\mathcal{D}^2)\cap D(\mathcal{B})$ and $f_n \in BUC(\mathbb{R}, E)$ such that $\|u_n - u\|_\infty \to 0$, $\|f_n - f\|_\infty \to 0$ and $\mathcal{D}^2 u_n - \mathcal{B}u_n = f_n$. Therefore, u_n are classical solutions of 5.3 (with f replaced by f_n), so that the following holds (for some x_n, y_n in E):

$$u_n(t) = u_n(0) + tu_n'(0) + A\int_0^t (t-s)u_n(s)ds + \int_0^t (t-s)f_n(s)ds,\ t \in \mathbb{R}. \tag{5.5}$$

From $\|u_n - u\|_\infty \to 0$ and $u_n', u_n'' \in BUC(\mathbb{R}, E)$, it follows that $u_n(0)$ and $u_n'(0)$ converge to some x and y, respectively. Using (5.5) and a standard argument involving closed operators, we obtain (5.4).

Conversely, assume that u satisfies (5.4). Choose $\phi_n \in L^1(\mathbb{R})$ such that $supp\,\hat{\phi}_n$ are compact and $u*\phi_n \to u$ as $n \to \infty$ uniformly (see e.g. [14], Chapter 6). Then we have

$$u*\phi(t) = \left[\int_{-\infty}^{\infty}\phi(t)dt\right]x + \left[\int_{-\infty}^{\infty}(t-s)\phi(s)ds\right]y+$$
$$A\int_0^t (t-s)(u*\phi)(s)ds + \int_0^t (t-s)(f*\phi)(s)ds,\ t\in\mathbb{R},$$

so that $u_n = u*\phi_n$ is a classical solution of (5.3), with f replaced by $f*\phi_n$. Thus, $u_n \in D(\mathcal{L}_0)$ and $\mathcal{L}_0 u_n = f_n \to f$, so that $u \in D(\mathcal{L})$ and $\mathcal{L}u = f$, i.e. u is a mild solution. ☐

Let Σ be the equation spectrum for equation

$$\mathcal{L}u = f, \tag{5.6}$$

where $\mathcal{L}$ is defined above.

Proposition 5.4. $\Sigma = \{\lambda \in \mathbb{R} : -\lambda^2 \in \sigma(A)\}$.

Proof. Assume that $\lambda_0 \in \mathbb{R}$ and $(\lambda_0^2 + A)$ is invertible. Then there exists a neighborhood $\mathcal{U}$ of λ_0 such that $(\lambda^2+A)^{-1}$ exists for every $\lambda \in \overline{\mathcal{U}}$. Consider $M(\overline{\mathcal{U}})$. The restriction $\mathcal{D}^2|M(\overline{\mathcal{U}})$ is a bounded operator and $\sigma(\mathcal{D}^2|M(\overline{\mathcal{U}})) = \{-\lambda^2 : \lambda \in \overline{\mathcal{U}}\}$. Therefore, by Lemma 3.7, $(\mathcal{D}^2 - \mathcal{B})|M(\overline{\mathcal{U}})$ is invertible, i.e. $\lambda_0 \notin \Sigma$. Thus we have showed $\Sigma \subseteq \{\lambda \in \mathbb{R} : (\lambda^2 + A)$ is not invertible$\}$ Conversely, let $\lambda_0 \notin \Sigma$. Then there exists a neighborhood $\mathcal{U}$ of λ_0 such that $M(\overline{\mathcal{U}})$ is regularly admissible. In particular, $M(\{\lambda_0\})$ is regularly admissble. Since $M(\{\lambda_0\}) = \{e^{i\lambda_0 t}x : x \in E\}$, it follows that for every $x \in E$ there exists a unique $y \in E$ such that the function $u(t) = e^{i\lambda_0 t}y$ is a mild solution of (5.3)-(5.6). This implies that for every $x \in E$ there exists a unique $y \in E$ such that $(-\lambda_0^2 - A)y = x$, i.e. $\lambda_0^2 + A$ is invertible. ☐

Therefore, we have the following theorems (cf. [1,23]).

Theorem 5.5. *Assume that $\sigma(A) \cap (-\infty, 0]$ is countable, f is almost periodic, and u is a uniformly continuous bounded mild solution of* (5.3). *Then $Sp_{ap}(u)$ is countable. In particular, u is almost periodic, provided one of the following conditions holds:*

(i) *$E \not\supset c_0$;*
(ii) *$u(\mathbb{R})$ is relatively weakly compact;*
(iii) *$u(t)$ is totally ergodic.*

Theorem 5.6. *Assume that $\sigma(A) \cap (-\infty, 0]$ is countable, f is asymptotically almost periodic, and u is a uniformly continuous bounded mild solution of* (5.3). *Then u is asymptotically almost periodic if (and only if) $u(t)$ is totally ergodic.*

5.3. **Convolution equations.** Consider the equation

$$\mu * u = f, \tag{5.7}$$

where μ is a complex bounded measure on $\mathbb{R}$, $f \in BUC(\mathbb{R}, E)$. This equation has the form

$$\mathcal{L}u = f, \tag{5.8}$$

where $\mathcal{L}$ is the convolution operator

$$Lu = \mu * u, \ u \in BUC(\mathbb{R}, E).$$

It follows from the Wiener Tauberian theorem that the equation spectrum Σ for (5.7) or (5.8) coincides with all λ such that $\hat{\mu}(\lambda) = 0$. Therefore, we have the following theorems (cf. [21], [22]).

Theorem 5.7. *Assume that the set $\Sigma = \{\lambda \in \mathbb{R} : \mu(\lambda) = 0\}$ is countable, f is almost periodic and u is a uniformly continuous bounded mild solution of* (5.7). *Then $Sp_{ap}(u)$ is countable. In particular, u is almost periodic provided one of the conditions (i)-(iii) in Theorem 5.1 holds.*

Theorem 5.8. *Assume that the set $\Sigma = \{\lambda \in \mathbb{R} : \mu(\lambda) = 0\}$ is countable, f is asymptotically almost periodic and u is a uniformly continuous bounded mild solution of* (5.7). *Then u is asymptotically almost periodic if (and only if) $u(t)$ is totally ergodic.*

5.4. **Volterra equations.** Consider the Volterra equation

$$u'(t) = Au(t) + \int_0^\infty dB(\tau)u(t-\tau) + f(t), \tag{5.9}$$

where A is a closed linear operator on a Banach space E with dense domain $D(A)$, $\{B(t)\}_{t\geq 0}$ is a family of closed linear operators in E with $D(B(t)) \supset D(A)$ for all $t \geq 0$ such that $B \in BV(\mathbb{R}_+, \mathbb{B}(Y, E))$ (the space of $\mathbb{B}(Y, E)$-valued functions of bounded variation over $\mathbb{R}_+$) and $Y = D(A)$ with the graph norm. (5.9) is associated with an operator equation $\mathcal{D}u - \mathcal{B}u = f$, where $\mathcal{D}f = f'$ and $\mathcal{B}$ is defined by

$$(\mathcal{B}u)(t) = Au(t) + \int_0^\infty dB(\tau)u(t-\tau), \ u \in BUC(\mathbb{R}, E).$$

The spectrum Σ of the equation $(\mathcal{D} - \mathcal{B})u = f$ is

$$\Sigma = \{\lambda \in \mathbb{R} : [i\lambda - A - \widehat{dB}(i\lambda)] \text{ is not invertible}\}.$$

Therefore, we obtain the following results (cf. [23]).

Theorem 5.9. *Assume that Σ is countable, f is almost periodic and u is a uniformly continuous bounded mild solution of* (5.9). *Then $Sp_{ap}(u)$ is countable. In particular, u is almost periodic provided one of the conditions (i)-(iii) in Theorem 5.1 holds.*

Theorem 5.10. *Assume that Σ is countable, f is asymptotically almost periodic and u is a uniformly continuous bounded mild solution of* (5.9). *Then u is asymptotically almost periodic if (and only if) $u(t)$ is totally ergodic.*

5.5. **Difference equations.** Consider the difference equation

$$u_{n+1} = Au_n + \sum_{k=1}^{m} B_k u_{n-k} + f_n, \tag{5.10}$$

where A is an arbitrary closed operator and $B_k, k = 1, 2, ..., m,$ are arbitrary bounded operators on a Banach space E. (5.10) has the form

$$\mathcal{L}u = f,$$

where $u = (u_n)$ and $f = (f_n)$ are elements (sequences) in $l^\infty(\mathbb{Z}, E)$ and $\mathcal{L}$ is a closed linear operator in $l^\infty(\mathbb{Z}, E)$ defined by

$$D(\mathcal{L}) := \{u = (u_n) : u_n \in D(A)\ \forall n\} \text{ and } (\mathcal{L}u)_n = u_{n+1} - Au_n - \sum_{k=1}^{m} B_k u_{n-k}.$$

Lemma 5.11. *The equation spectrum* Σ *of* (5.10) *is given by*

$$\Sigma = \left\{ \lambda \in \Gamma : (\lambda - A - \sum_{k=1}^{m} \lambda^{-k} B_k) \textit{ is not invertibe} \right\}.$$

Proof. Assume that $\lambda_0 \in \Gamma$ is such that $(\lambda_0 - A - \sum_{k=1}^{m} \lambda_0^{-k} B_k)$ is invertible. Then there exists a neighborhood $\mathcal{U}$ of λ_0 such that $(\lambda - A - \sum_{k=1}^{m} \lambda^{-k} B_k)^{-1}$ exists for all $\lambda \in \mathcal{U}$. Let Λ be a neighborhood of λ_0 such that $\overline{\Lambda} \subset \mathcal{U}$. We show that $M(\overline{\Lambda})$ is regularly admissible, i.e. for every $f \in M(\overline{\Lambda})$, there exists a unique solution $u \in M(\overline{\Lambda})$ of (5.10).

First, we show the uniqueness. Assume that u^1 and u^2 are solutions in $M(\overline{\Lambda})$ of (5.10) with $f \in M(\overline{\Lambda})$. Then $u = u^1 - u^2$ is a solution in $M(\overline{\Lambda})$ of the homogeneous equation

$$u_{n+1} = Au_n + \sum_{k=1}^{m} B_k u_{n-k}.$$

By considering the Carleman transform of both parts of the above equation, and denoting the shift operator by S ($(Su)_n = u_{n+1}$), we have

$$\begin{aligned}\widetilde{(Su)}(\lambda) &= \lambda^{-1}\tilde{u} - \lambda^{-1}u_0 \\ &= A\tilde{u}(\lambda) + \sum_{k=1}^{m} \lambda^k B\tilde{u}(\lambda) + \sum_{k=1}^{m} \lambda^k \sum_{j=1-k}^{0} \lambda^{j-1} x_j,\end{aligned}$$

or

$$(\lambda^{-1} - A - \sum_{k=1}^{m} \lambda^k B)\tilde{u}(\lambda) = y, \tag{5.11}$$

for some $y \in E$. From

$$\mu - A - \sum_{k=1}^{m} \mu^{-k} B_k = \lambda - A - \sum_{k=1}^{m} \lambda^{-k} B_k + (\mu - \lambda) - \sum_{k=1}^{m} (\mu^k - \lambda^k) B_k$$

it follows that if $(\lambda - A - \sum_{k=1}^{m} \lambda^{-k} B_k)$ is invertible and $|\mu - \lambda| < \epsilon$ for sufficiently small $\epsilon > 0$, then $(\mu - A - \sum_{k=1}^{m} \mu^{-k} B_k)$ is invertible. Therefore, the set

$$\Omega := \{\lambda : (\lambda - A - \sum_{k=1}^{m} \lambda^{-k} B_k) \text{ is invertible}\}$$

is open. From (5.11) it follows that if $(\xi - A - \sum_{k=1}^{m} \xi^{-k} B_k)^{-1}$ exists for some ξ from a neighborhood of λ_0, then $\tilde{u}(\lambda)$ has analytic continuation in a neighborhood of λ_0. Thus, we have shown that

$$Sp(u) \subseteq \{\lambda : (\lambda - A - \sum_{k=1}^{m} \lambda^{-k} B_k) \text{ is not invertible}\}.$$

On the other hand, $Sp(u) \subseteq \overline{\Lambda}$ and $(\lambda - A - \sum_{k=1}^{m} \lambda^{-k} B_k)$ is invertible for all $\lambda \in \overline{\Lambda}$. Therefore, $Sp(u) = \emptyset$, hence $u = 0$. The uniqueness is proved.

Now we prove the existence of a solution $u \in M(\overline{\Lambda})$. Suppose $f \in M(\overline{\Lambda})$. Let $\phi \in l^1(\mathbb{Z})$ be such that $supp\, \hat{\phi} \subset \mathcal{U}$ and $\hat{\phi} = 0$ on a neighborhood of $\overline{\Lambda}$. Define

$$G(n) = \int_\Gamma (\lambda - A - \sum_{k=1}^{m} \lambda^{-k} B_k)^{-1} \hat{\phi}(\lambda) \lambda^{-n} d\lambda,$$

(where the integral is in fact over $\mathcal{U}$, since $supp\, \hat{\phi} \subset \mathcal{U}$). Double integration by parts yields

$$G(n+1) = \frac{1}{n^2} \int_\Gamma \left[\frac{d^2}{d\lambda^2} (\lambda - A - \sum_{k=1}^{m} \lambda^{-k} B_k)^{-1} \hat{\phi}(\lambda) \right] \lambda^{-n+2} d\lambda,$$

which implies that $\sum_{n=-\infty}^{\infty} \|G_n\| < \infty$. Therefore, we can define $u \in l^\infty(\mathbb{Z}, E)$ by

$$u_n = \sum_{k=-\infty}^{\infty} G(n-k) f_k, \ n \in \mathbb{Z}.$$

From the uniqueness theorem it follows that u is a solution of (5.10) and from spectral properties of sequences it follows that $Sp(u) \subseteq Sp(f) \subseteq \Lambda$. Thus, we have proved the inclusion $\Sigma \subseteq \{\lambda \in \Gamma : (\lambda - A - \sum_{k=1}^{m} \lambda^{-k} B_k) \text{ is not invertibe}\}$. To show the inverse inclusion, suppose that $\lambda_0 \notin \Sigma$. Then there exists an open neighborhood $\mathcal{U}$ of λ_0 such that $M(\overline{\mathcal{U}})$ is regularly admissible. This implies that $M(\{\lambda_0\})$ is admissible, i.e. for every $f_n = (\lambda_0^n x)$, $x \in E$, there exists a unique solution $u = (\lambda_0^n y)$ of (5.10), for some $y \in E$. Therefore, for every $x \in E$ there exists a unique $y \in E$ such that $(\lambda_0 - A - \sum_{k=1}^{m} \lambda_0^{-k} B_k) y = x$, which implies that $(\lambda_0 - A - \sum_{k=1}^{m} \lambda_0^{-k} B_k)$ is invertible. □

Thus, we obtain the following theorems.

Theorem 5.12. *Assume that* $\Sigma := \left\{ \lambda \in \Gamma : (\lambda - A - \sum_{k=1}^{m} \lambda^{-k} B_k) \textit{ is not invertible} \right\}$ *is countable,* $f = (f_n)$ *is almost periodic and* $u = (u_n)$ *is a bounded solution of* (5.10)*. Then* $Sp_{ap}(u)$ *is countable. In particular,* u *is an almost periodic sequence provided one of the following conditions holds:*

(i) $E \not\supset c_0$*;*

(ii) u *has relatively compact range;*

(iii) (u_n) *is totally ergodic.*

Theorem 5.13. *Assume that Σ is countable, $f = (f_n)$ is an asymptotically almost periodic sequence and u is a uniformly continuous bounded mild solution of* (5.10)*. Then u is an asymptotically almost periodic sequence if (and only if) $u(t)$ is totally ergodic.*

In conclusion, we note the results on discrete semigroups in Section 4 can be used to study asymptotic properties of ω-periodic functional-differential equations, since these equations are characterized by commutativity of the corresponding operator $\mathcal{L}$ with $T(\omega)$, the translation by ω. These applications to periodic equations will be treated in a subsequent paper.

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