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MULTI-PRODUCT SUPPLY DEMAND NETWORKS WITH ELEMENTARY FLOWS

DINH THE LUC

Dedicated to Tran Duc Van on the occasion of his sixtieth birthday

Abstract. In this paper, we study a multi-product multi-criteria supply demand network with capacity constraints. We analyze different concepts of equilibrium and establish some relationships between them. Particular attention is paid on elementary flows and on the construction of variational inequalities which are equivalent to network equilibrium problems.

1. Introduction

In recent years, multi-product multi-criteria supply demand networks have become a subject of intensive study. This is because such networks find abounding applications in several areas of applied sciences such as internet communications, transport, economics, decisions etc. (see [1–3, 5, 7, 10–13, 15]). The concept of equilibrium of transport networks initially introduced by Wardrop [14] in 1952 is considered as a starting point of supply demand network investigation. In contrast to a single-product network, the concept of equilibrium for multi-product multi-criteria networks is not unique, depending on how to interpret Wardrop's equilibrium conditions when the product bundles in a flow and their associated costs are vectorial. In the existing literature, a number of interpretations are known which lead to different concepts of multi-product equilibrium, but their relationship is not fully exploited and sometimes misunderstood. For this reason, we wish to develop a general approach to equilibrium for multi-product networks. We prove equivalence of multi-product equilibrium and solutions of associated variational inequalities. The concept of elementary flows introduced in [10] plays a major role in the present work.

The paper is structured as follows. Section 2 describes a network model to study with a matrix formula linking arc flows and path flows. By focusing on networks without capacity constraints we discuss different concepts of equilibrium and point out some inadequacies in the current literature on these concepts. In Section 3, we deal with most important concepts of equilibrium in

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multi-product networks with capacity constraints and establish some relationships between them. Section 4 is devoted to elementary flows. We show that every feasible flow can be decomposed into a sum of a pattern flow and elementary flows, which allow us to treat a network with elementary flows only. In the final section, we construct three variational inequality problems which are equivalent to the problems of finding weak, strong and ideal equilibria of the network. As a particular case we illustrate that the variational inequality problem over all feasible flows studied in [2] for networks without capacity constraints can easily be derived from the variational inequality problem over elementary flows, which is associated to ideal equilibrium.

2. Multi-product single-criterion supply demand network

We begin the section by describing a network model to study. A supply-demand network $G = [N, A, W]$ consists of a set of nodes N, a set of n directed arcs $A = \{a_1, ..., a_n\}$ and a set W of origin destination pairs of nodes $w = (x, x')$ with $x, x' \in N$ such that there is a path from x to x'. For a pair of nodes $w = (x, x')$, the set of available paths from the origin x to the destination x' is denoted by P_w, and the set of all available paths of the network is denoted by $P = \{p_1, ..., p_m\} = \cup_{w \in W} P_w$.

In a multi-product or multi-class model, it is assumed that there are q different kinds of products to traverse the network. Given a path $p_j \in P$, let $y_{ji} \in \mathbb{R}$ denote the amount of the ith product to be transported on the path p_j. The matrix $Y = (y_{ji})_{m \times q}$ is called a path flow in the network. Thus, each row vector $Y_j = (y_{j1}, ..., y_{jq})$ of the matrix Y represents the vector of the q products to traverse the path P_j, while the column vector $Y^i = (y_{1i}, ..., y_{mi})^T$ (here $(\cdot)^T$ denotes the transpose) represents the vector of the ith product to traverse the m paths of the network.
To evaluate the transportation of products in the network a cost function C is given in form of an $m \times q$ matrix $C(Y) = (c_{ji}(Y))_{m \times q}$. In a single-criterion network, the entries $c_{ji}(Y)$ are real numbers, and in a multi-criteria network they are vectors of a multi-dimensional space, say $\mathbb{R}^\ell$ with $\ell > 1$. The jth row of entries $C_j(Y) = (c_{j1}(Y), ..., c_{jq}(Y))$ represents the cost for the path p_j, and the ith column $C^i(Y) = (c_{1i}(Y), ..., c_{mi}(Y))^T$ represents the cost concerning the ith product on the paths $p_1, ..., p_m$. For every origin destination pair $w \in W$, the index set $J(w)$ consists of all $j \in \{1, ..., m\}$ such that $p_j \in P_w$, and the set $C(w)$ consists of all vectors $C_j(Y)$ with $j \in J(w)$.

Sometimes arc flows are also considered in association with path flows. If z_{ki} denotes the amount of the ith product to be transported on the arc a_k, then the matrix Z whose entries are $z_{ki}, k = 1, ..., n$ and $i = 1, ..., q$ represents an arc flow in the network. A vector-valued cost function for the arc flow Z is given by a matrix $\hat{C}(Z)$ with entries $\hat{c}_{ki}(Z), k = 1, ..., n$ and $j = 1, ..., q$. It is known that given a path flow Y, an associated arc flow Z can be determined by the formula

$$Z = \Delta Y,$$

where Δ is the so-called incident matrix whose entries δ_{kj} are given by

$$\delta_{kj} = \begin{cases} 1 & \text{if } a_k \in p_j, \\ 0 & \text{otherwise.} \end{cases}$$

The cost functions of the arc flow Z and the path flow Y are then linked by the following matrix equality:

$$C(Y) \quad = \quad \Delta^T \hat{C}(Z).$$

From now on, we fix a path flow $\bar{Y}$ and write C and c_{ji} instead of $C(\bar{Y})$ and $c_{ji}(\bar{Y})$ if no misunderstanding occurs. We assume further that a positive demand function $d_{wi}(\bar{Y})$ is given which expresses the quantity of the ith product to be transported from the origin x to the destination x' of the pair $w = (x, x') \in W$, and that the demand vector $d_w = (d_{w1}(\bar{Y}), ..., d_{wq}(\bar{Y}))$ is non null. The lower and upper capacity constraints on each product i and on each path p_j are respectively l_{ji} and $u_{ji} \in \mathbb{R}$ with $l_{ji} < u_{ji}$. If lower and upper capacity constraints are given on arcs, say $l(k, i)$ and $u(k, i)$ for $i = 1, ..., q$ and $k = 1, ..., n$, then lower and upper capacity constraints on a path p_j are defined respectively by $l_{ji} = \max\{l(k, i) : a_k \in P_j\}$ and $u_{ji} = \min\{u(k, i) : a_k \in P_j\}$.
We say that a path flow Y is feasible if it satisfies the capacity constraints and the conservation of flow equations:

$$l_{ji} \leqq y_{ji} \leqq u_{ji} \qquad \text{for all } i = 1, ..., q; j = 1, ..., m \tag{2.1}$$

$$\textstyle\sum_{j \in J(w)} y_{ji} = d_{wi}(\bar{Y}) \quad \text{for all } i = 1, ..., q; w \in W. \tag{2.2}$$

The set of all feasible flows is denoted by $K(\bar{Y})$. When $l_{ji} = 0$ and $u_{ji} = \infty$ for all i and j the network is called without capacity constraints.
In the space $\mathbb{R}^q$ we distinguish the following order relations: strict inequality "$>$" is understood as "componentwise strictly greater than", and inequality " $\geq$ " means "componentwise greater than or equal to" and not equal to. The binary relations "$>$" and "$\geq$" are actually partial orders generated by the positive orthant $\mathbb{R}^q_+$ of the space $\mathbb{R}^q$. Namely, for two vectors c and c' from $\mathbb{R}^q$, one has $c \geq c'$ (respectively $c > c'$) if and only if $c - c' \in \mathbb{R}^q_+ \setminus \{0\}$ (respectively $c - c' \in \operatorname{int} \mathbb{R}^q_+$), where int $\mathbb{R}^q_+$ is the interior of $\mathbb{R}^q_+$. The relation " $\geqq$ " means either "$\geq$" or "$=$".
Given a set $D \subseteq \mathbb{R}^q$, the infimum of D, denoted by Inf(D), is the vector whose ith component is the infimum of the projection of D on the ith axis. If this infimum is finite and belongs to the set D, it is called the ideal minimal element of D. An element d of D is called minimal (respectively weak minimal) if there is no other element $d' \in D$ such that $d \geq d'$ (respectively $d > d'$). The sets of all minimal and weak minimal elements of D are denoted by Min(D) and WMin(D) respectively. The cone generated by the set D is denoted by cone(D), that is cone(D) $= \{td : d \in D, t \geqq 0\}$, and its closure is denoted by clcone(D).

In the remainder of this section, we wish to compare different concepts of equilibrium. We restrict ourselves to the case of multi-product, single-criterion network flows without capacity constraints for the sake of simplicity and compatibility with existing definitions we meet in the literature. Consider the following conditions/ implications:

(H1) for every $w \in W$ and $j \in J(w)$,

$$\bar{Y}_j \geq 0 \quad \Rightarrow \quad C_j = \mathrm{Inf}\,(C(w)),$$
$$\bar{Y}_j = 0 \quad \Rightarrow \quad C_j \geqq \mathrm{Inf}\,(C(w));$$

(H2) for every $w \in W$ and $j, k \in J(w)$,

$$[(C(w) - C_k) \cap (-\mathbb{R}^q_+) = \{0\}, C_j - C_k \neq 0] \quad \Rightarrow \quad \bar{Y}_j = 0;$$

(H3) for every $w \in W$ and $j, k \in J(w)$,

$$[\mathrm{clcone}(C(w) + \mathbb{R}^q_+ - C_k) \cap (-\mathbb{R}^q_+) = \{0\}, C_j - C_k \neq 0] \quad \Rightarrow \quad \bar{Y}_j = 0;$$

(H4) for every $w \in W$ and $j, k \in J(w)$,

$$C_j - C_k \geq 0 \quad \Rightarrow \quad \bar{Y}_j = 0;$$

(H5) for every $w \in W$ and $j \in J(w)$,

$$C_j \notin \mathrm{Min}\,(C(w)) \quad \Rightarrow \quad \bar{Y}_j = 0; \text{and}$$

(H6) for every $w \in W$, $j, k \in J(w)$, and $i = 1, ..., q$,

$$c_{ji} - c_{ki} > 0 \quad \Rightarrow \quad \bar{y}_{ji} = 0.$$

Condition (H1) was recently introduced by Cheng and Wu [2] and the pattern flow $\bar{Y}$ satisfying it is called a Wardrop equilibrium. This equilibrium is a direct extension of the classical equilibrium principle formulated by Wardrop in [14] for a single-product single-criterion network. Notice that the second implication of (H1) is superfluous because for any path $p_j \in P_w$ inequality $C_j \geqq \mathrm{Inf}\,(C(w))$ is always true.

The authors of [2] have stated that $\bar{Y}$ is a Wardrop equilibrium if and only if it satisfies (H4) (see [2, Proposition 2.1]). This assertion is, however, not always true as it will be clear from Proposition 2.2 below and from Example 2.3. As a matter of fact conditions (H1) and (H4) lead to different concepts of equilibrium whenever the multiplicity of products for transport in the network is present. Condition (H4) has been studied in [2] for multi-criteria networks, and mentioned by Raciti in [13] as the strong vector Wardrop equilibrium. Conditions involving WMax$(C(w))$ instead of Max$(C(w))$ are possible and handled in a similar way.

Condition (H3) has been introduced by Wu and Cheng in [15] to define the so-called Benson equilibrium which is a version of Benson proper efficient solutions of vector optimization problems. As we shall see in Proposition 2.2, the operation of taking closed cone in (H3) is unnecessary and in reality (H2) and (H3) are equivalent.

Condition (H6) has been originally developed by Li, Teo and Yang in [7] for networks with capacity constraints. We shall give more discussion on it in the next section. The following lemma is standard.

Lemma 2.1. *Let D be a finite subset of $\mathbb{R}^q$ and $d \in D$. Then the following relations are equivalent*

$$\operatorname{clcone}(D + \mathbb{R}^q_+ - d) \cap (-\mathbb{R}^q_+) = \{0\} \tag{2.3}$$

$$(D - d) \cap (-\mathbb{R}^q_+) = \{0\}. \tag{2.4}$$

Proof. The implication (2.3) $\Rightarrow$ (2.4) is clear because the set $D - d$ is a subset of $\operatorname{clcone}(D + \mathbb{R}^q_+ - d)$ and the origin of the space belongs to both of them. For the converse suppose the contrary that (2.4) is true, but (2.3) is not. There is a nonzero vector a belonging to the intersection on the left hand side of (2.3), say

$$a = \lim_{\alpha \to \infty} t_\alpha (d_\alpha - d + u_\alpha) \tag{2.5}$$

for some positive numbers t_α, some vectors d_α from D and $u_\alpha \in \mathbb{R}^q_+$. Since D is a finite set, we may assume without loss of generality that $d_\alpha = d_0$ for some $d_0 \in D$. If $d_0 - d = 0$, we arrive at a contradiction that $a \in \mathbb{R}^q_+ \cap (-\mathbb{R}^q_+)$ and $a \neq 0$. It remains to consider the case $d_0 - d \neq 0$. We may also assume that the sequence $\{t_\alpha\}_\alpha$ converges to some limit t among three possible values: 1) $t = 0$, 2) $t = \infty$, and 3) $t \in (0, \infty)$. In the first case, $a = \lim_{\alpha \to \infty} t_\alpha u_\alpha \in \mathbb{R}^q_+$, which contradicts the hypothesis. In the second case, it follows from (2.5) that $a = t_\alpha(d_\alpha - d + u_\alpha) + o(t_\alpha)$ with $\lim_{\alpha \to \infty} o(t_\alpha)/t_\alpha = 0$. By dividing the latter equality by t_α and passing to the limit as α tends to ∞, we obtain $d_0 - d = -\lim_{\alpha \to \infty} u_\alpha \in -\mathbb{R}^q_+ \setminus \{0\}$ which contradicts (2.4). In the case 3), a similar argument yields

$$d_0 - d = \frac{a}{t} - \lim_{\alpha \to \infty} u_\alpha \in -\mathbb{R}^q_+ \setminus \{0\}$$

which is a contradiction too. $\square$

We remark that the conclusion of Lemma 2.1 remains true under a milder condition on D. For instance, when D is not finite, but the set $\operatorname{cone}(D - d)$ has a compact base, which means that there is a compact set B not containing the origin of the space such that $\operatorname{cone}(D - d) = \operatorname{cone}(B)$, then the argument of the proof above goes through. In particular, the conclusion of Lemma 2.1 is true when D is a polyhedral set. Here are some relationships between (H1)-(H6).

Proposition 2.2. *Given a feasible pattern flow $\bar{Y}$ on the network G. The following assertions hold:*

(i) (H1) $\Leftrightarrow$ (H2) $\Leftrightarrow$ (H3). *Each of these conditions implies that for every $w \in W$, the set $C(w)$ has ideal minimal elements. Moreover, under the latter condition on $C(w)$, all conditions* (H1) *through* (H5) *are equivalent.*

(ii) (H4) $\Leftrightarrow$ (H5).

(iii) (H1) $\Rightarrow$ (H6). *The converse* (H6) $\Rightarrow$ (H1) *is true provided $q = 1$.*

Proof. We note that for every $w \in W$, the set $C(w)$ is finite, hence in view of Lemma 2.1 conditions (H2) and (H3) are equivalent. To prove the first part of (i), it suffices to establish equivalence between (H1) and (H2). We assume (H1). Since for each $w \in W$ the demand vector d_w is non null, there must be some path $p_{k_0} \in P_w$ on which the flow $\bar{Y}_{k_0}$ is non null. Hence the cost C_{k_0} is an ideal

minimal element of $C(w)$. Let $k \in J(w)$ satisfy $(C(w) - C_k) \cap (-\mathbb{R}^q_+) = \{0\}$. Then $C_k = C_{k_0} = \mathrm{Inf}\,(C(w))$, and $C_j \geq \mathrm{Inf}\,(C(w))$ for every $j \in J(w)$ with $C_j - C_k \neq 0$. By (H1), $\overline{Y}_j = 0$, which shows that (H2) holds. Now assume (H2). Since the set $C(w)$ is finite, it has minimal elements. Let C_k be one of them. Then $(C(w) - C_k) \cap (-\mathbb{R}^q_+) = \{0\}$. For any $j \in J(w)$, if C_j is not minimal, then $C_j \neq C_k$ and by (H2), the corresponding flow $\bar{Y}_j$ is null. If C_j is minimal, but $C_j \neq C_k$, then we also have $\bar{Y}_j = 0$ by (H2). With C_j minimal, switching the roles of C_j and C_k we obtain $\bar{Y}_k = 0$ too. Thus, if the set $\mathrm{Min}(C(w))$ consists of more than two elements, the flow $\bar{Y}$ is null on every path joining w, which is impossible because the demand is not null. Consequently, the set $\mathrm{Min}(C(w))$ has only one value, say C_*. We deduce $C_j \geqq C_*$ for all $j \in J(w)$, which shows that C_* is the ideal minimal element of $C(w)$ and (H1) follows.
For the second part of (i), assume that for every $w \in W$, the set $C(w)$ has ideal minimal elements. It suffices to prove equivalence between (H1) and (H4), because the equivalence between (H4) and (H5) will be given in (ii). Let $j, k \in J(w)$ satisfy $C_j - C_k \geq 0$. Then C_j is not ideal minimal. Under (H1), one has $\bar{Y}_j = 0$ and obtains (H4). Conversely, if (H4) holds and if $\bar{Y}_j \geq 0$, then C_j must be ideal minimal, which shows that (H1) is true. Indeed, if C_j were not ideal minimal, there would exist some ideal element C_k such that $C_j \geq C_k$ which yields $\bar{Y}_j = 0$, a contradiction. By this, (H4) is equivalent to (H1).
We proceed to (ii) by assuming (H4). Let $C_j \notin \mathrm{Min}(C(w))$. By definition, there is some $C_k \in C(w)$ such that $C_j \geq C_k$. In view of (H4) one has $\bar{Y}_j = 0$ and (H5) follows. Conversely, if (H5) holds and if $C_j - C_k \geq 0$ for some $j, k \in J(w)$, then C_j is not a minimal element of $C(w)$ and in view of (H5) the flow $\bar{Y}_j$ is null. Thus, (H4) is true and we obtain the equivalence between (H4) and (H5).
Finally, suppose (H1). Strict inequality $c_{ji} > c_{jk}$ for some $i, k \in \{1, ..., q\}$ and $j \in J(w)$ in (H6) implies that C_j is not an ideal minimal element of $C(w)$. By (H1), one has $\bar{Y}_j = 0$. In particular, $\bar{y}_{ji} = 0$ and (H6) follows. When $q = 1$ inequality $\bar{Y}_j \geq 0$ means $\bar{y}_{j1} > 0$, and so under (H6) one has $c_{j1} - c_{k1} \leqq 0$ for all $k \in J(w)$, that is $c_{j1} = \mathrm{Inf}\,(C(w))$. Thus, for $q = 1$, conditions (H1) and (H6) are equivalent. □

We note that (H6) treats individually the products to traverse within the network, and so its study belongs to single-product supply demand models. Another remark is the fact that when the set $C(w)$ has no ideal minimal elements, the implication (H4) ⇒ (H1) may fail as it is shown by the next example. Proposition 2.1 of [2] and Proposition 3.2 of [15] are then not always available.

Example 2.3. Consider a network consisting of four nodes $\{N_i : i = 1, ..., 4\}$, one origin destination pair $w = (N_1, N_4)$ and two paths p_1 and p_2 connecting w via N_2 and N_3 respectively. We assume there are two products in the network. Let a feasible pattern flow Y be given by its rows $Y_1 = (20, 320)$ and $Y_2 = (10, 500)$ representing the quantities of the two products to traverse the paths p_1 and p_2 respectively. Assume further that the cost matrix associated to the path flow Y has its rows $C_1 = (2, 16)$ and $C_2 = (1, 25)$. The infimum of $C(w)$ is the vector

$(1, 16)$. It is clear that (H1) does not hold for the pattern flow Y because both vectors Y_1 and Y_2 are positive and no cost vector is equal to $\operatorname{Inf}(C(w))$. However, (H4) does hold, simply because the cost vectors C_1 and C_2 are not comparable.

In multi-product networks, equilibria defined via (H4) and (H6) do not follow from each other. The flow given in the previous example satisfies (H4), but not (H6). The example below shows that (H6) does not imply (H4) either.

Example 2.4. Consider the network of the previous example. Let a feasible pattern flow Y be given by its rows $Y_1 = (0, 830)$ and $Y_2 = (30, 0)$ representing the quantities of the two products to traverse the paths p_1 and p_2 respectively. Assume further that the cost matrix associated to the path flow Y has its rows $C_1 = (2, 16)$ and $C_2 = (2, 25)$. Then (H6) holds, but not (H4) because $C_2 \geq C_1$ with $Y_2 \neq 0$.

3. Multi-product multi-criteria supply demand network with capacity constraints

In this section, we analyze a multi-product multi-criteria supply demand network G with capacity constraints in which the costs $c_{ji}(Y)$ take values in $\mathbb{R}^\ell$ with $\ell > 1$. This model has been extensively studied in recent years (see [7, 10] and many references given therein). For $w \in W$, the demand vector $(d_{w1}, ..., d_{wq})$ is denoted by d_w, and for a path p_j, the upper and lower capacity bound vectors $(u_{j1}, ..., u_{jq})$ and $(l_{j1}, ..., l_{jq})$ are respectively denoted by U_j and L_j. It is common to impose the following restrictions on the demand

$$\sum_{j\in J(w)} L_j \leqq d_w \leqq \sum_{j\in J(w)} U_j \ \text{ for all } w \in W.$$

Otherwise, the network would have no feasible flows. Moreover, if either of equalities holds in the above restrictions, then the network has a unique feasible flow on the paths linking w. This case is not interesting from the mathematical point of view. Therefore, from now on we assume

$$\sum_{j\in J(w)} L_j \leq d_w \leq \sum_{j\in J(w)} U_j \ \text{ for all } w \in W. \tag{3.1}$$

Consider the following extensions of (H1) for a feasible pattern flow $\bar{Y}$:

(H7) for every $w \in W$ and $j \in J(w)$,

$$C_j \geq \operatorname{Inf} C(w) \quad \Rightarrow \quad \bar{Y}_j = L_j;$$

(H8) for every $w \in W$ and $j \in J(w)$,

$$C_j \geq \operatorname{Inf} C(w) \quad \Rightarrow \quad \text{either } \bar{Y}_j = L_j \text{ or } \bar{Y}_k = U_k \text{ for all } k \in J(w) \text{ with } C_k = \operatorname{Inf} C(w);$$

(H9) for every $w \in W$ and $j \in J(w)$,

$$C_j \geq \operatorname{Inf} C(w) \quad \Rightarrow \quad \text{either } \bar{Y}_j = L_j \text{ or } \bar{Y}_k = U_k \text{ for some } k \in J(w) \text{ with } C_k = \operatorname{Inf} C(w).$$

Needless to say that when the capacity constraints are absent, the above conditions reduce to (H1) of Section 2. Moreover, (H7) implies (H8) and in its turn (H8) implies (H9), but the converse is not true in general. The following proposition shows that these conditions are related to the existence of ideal minimal costs as in the case of networks without capacity constraints.

Proposition 3.1. *If the feasible pattern flow $\bar{Y}$ satisfies either of* (H7)*,* (H8) *and* (H9)*, then for every origin destination pair $w \in W$ the set of vector costs $C(w)$ has ideal minimal elements.*

Proof. Due to the implications of (H7), (H8) and (H9) we have mentioned, it suffices to prove the proposition when the flow $\bar{Y}$ satisfies (H9). Suppose to the contrary that for some origin destination pair $w \in W$ the set $C(w)$ has no ideal elements. This means that $C_j \geq \operatorname{Inf} C(w)$ for all $j \in J(w)$. In view of (H9), we have $\bar{Y}_j = L_j$. Summing up $\bar{Y}_j$ over all paths p_j joining w, we obtain

$$d_w = \sum_{j \in J(w)} \bar{Y}_j = \sum_{j \in J(w)} L_j$$

which contradicts (3.1). □

Since in most situations, ideal elements of a set of vectors do not exist, conditions (H7), (H8) and (H9) are very difficult to be fulfilled. Instead, extensions of (H4) offer a better choice for equilibrium in multi-product multi-criteria models with capacity constraints. We consider the following condition:

(H10) for every $w \in W$ and $j, k \in J(w)$,

$$C_j \geq C_k \quad \Rightarrow \quad \text{either } \bar{Y}_j = L_j \text{ or } \bar{Y}_k = U_k,$$

and its weaker version

(H11) for every $w \in W$ and $j, k \in J(w)$,

$$C_j \geq C_k \quad \Rightarrow \quad \text{either } \bar{Y}_j \not> L_j \text{ or } \bar{Y}_k \not< U_k.$$

In a model without capacity constraints, condition (H10) collapses to (H4) of Section 2. Condition (H11) is known as a necessary condition for a vector variational equilibrium introduced by Oettli in [12]. It is also named as a weak vector Wardrop principle in [13]. As far as we know, equilibrium under conditions (H7)-(H11) has not been developed for networks with capacity constrains. For these networks, the notion of equilibrium introduced by Li, Teo and Yang has received a lot of attention (see [7,8,10] for instance). However, it does not really take into account the multi-dimensionality of the products circulating within the network. Indeed, according to Definition 2.1 [7], a feasible pattern flow $\bar{Y}$ is said to be a vector equilibrium if for every $i = 1, ..., q$, $w \in W$ and $p_\beta, p_\alpha \in P_w$ one has implication

$$c_{\alpha i} \geq c_{\beta i} \Rightarrow \text{ either } \bar{y}_{\beta i} = u_{\beta i} \text{ or } \bar{y}_{\alpha i} = l_{\alpha i}. \tag{3.2}$$

In this equilibrium, the products $i = 1, ..., q$ are considered individually, without any link between them. In other words, study of this kind of equilibrium is within the framework of single-product multi-criteria networks with capacity constraints. In a similar vein, Raciti [13] studies equilibrium for a model without capacity constraints by requiring that for every $s \in \{1, ..., \ell\}$, $w \in W$ and $p_\beta, p_\alpha \in P_w$ one has implication

$$c^s_{\alpha i} > c^s_{\beta i} \;\Rightarrow\; \bar{y}_{\alpha i} = 0 \text{ for all } i = 1, ..., q. \tag{3.3}$$

In this definition not only the products are considered individually, but the criteria too. So its study belongs to the category of single-product single-criterion network equilibria.

Proposition 3.2. *Let $\bar{Y}$ be a feasible pattern flow. The following assertions hold.*

(i) (H7) $\Rightarrow$ (H10) $\Rightarrow$ (H11).
(ii) (H10) $\Rightarrow$ (H7) *provided that for every $w \in W$, the set $C(w)$ has ideal minimal elements and that*

$$d_w \not\geq \sum_{k \in J(w): C_k = \operatorname{Inf} C(w)} U_k + \sum_{j \in J(w): C_j \neq \operatorname{Inf} C(w)} L_j.$$

Proof. The implication (H10) $\Rightarrow$ (H11) is obvious. For the implication (H7) $\Rightarrow$ (H10) let $C_j \geq C_k$ for some $j, k \in J(w)$. Then C_j is not ideal minimal, and $\bar{Y}_j = L_j$ by (H7). This shows that (H10) is satisfied.
To prove (ii), we assume (H10). Let $C_j \geq \operatorname{Inf} C(w)$ for some $j \in J(w)$. Picking any $C_k = \operatorname{Inf} C(w)$, we obtain $C_j \geq C_k$ which implies that either $\bar{Y}_j = L_j$ or $\bar{Y}_k = U_k$. If $\bar{Y}_j = L_j$, we obtain (H7). If not, $\bar{Y}_s = U_s$ for all $s \in J(w)$ with $C_s = \operatorname{Inf} C(w)$. Consequently,

$$\begin{aligned} d_w &= \sum_{j \in J(w)} \bar{Y}_j \\ &= \sum_{k \in J(w): C_k = \operatorname{Inf} C(w)} U_k + \sum_{j \in J(w): C_j \neq \operatorname{Inf} C(w)} \bar{Y}_j \\ &\geq \sum_{k \in J(w): C_k = \operatorname{Inf} C(w)} U_k + \sum_{j \in J(w): C_j \neq \operatorname{Inf} C(w)} L_j \end{aligned}$$

which contradicts the hypothesis. □

It is not difficult to see that the converse of the implications in (i) is not true without additional hypotheses. The results of Propositions 3.1 and 3.2 suggest to call a pattern flow satisfying (H7), (H10) and (H11) as an ideal equilibrium, a strong equilibrium and a weak equilibrium respectively. Because a multi-product network hardly possesses ideal equilibrium flows, the concept of strong equilibrium seems to be most appropriate for multi-product networks. Weak equilibrium is particularly interesting in networks in which products are transported by bundles. For instance, machines sending from a factory to a destination are accompanied by a number of accessories. It is possible that on a path lower limits for certain accessories are reached while lower limits for other accessories are not. In such a model, strong equilibria infrequently exist and weak equilibria turn to be good substitutes.

4. Elementary flows

We proceed to define elementary flows that will be essential in constructing variational inequalities for a multi-product multi-criteria network with capacity constraints. For technical reasons, we allow flows with negative entries. Of course these are not feasible.

Definition 4.1. A path flow V of the network G is said to be elementary if there are some origin destination pair $w \in W$ and paths $p_\alpha, p_\beta \in P_w$ such that

$$\begin{aligned} V_\alpha &= -V_\beta \\ V_j &= 0 \qquad \text{for } j \in \{1, ..., m\} \setminus \{\alpha, \beta\}. \end{aligned}$$

This definition is a modification of elementary flows introduced in [10] in which products are considered separately. Let us establish some properties of elementary flows. Throughout we fix a feasible pattern flow $\overline{Y}$ and set

$$K_0(\overline{Y}) = \{Y \in K(\bar{Y}) : Y - \overline{Y} \text{ is elementary}\}$$

$$K_+(\bar{Y}) = \{Y \in K(\bar{Y}) : Y - \overline{Y} \text{ is elementary with } (Y - \bar{Y})_\alpha \geq 0 \text{ for some } \alpha\} \cup \{\bar{Y}\}$$

$$K'_+(\bar{Y}) = \{Y \in K(\bar{Y}) : Y - \overline{Y} \text{ is elementary with } (Y - \bar{Y})_\alpha > 0 \text{ for some } \alpha\} \cup \{\bar{Y}\}.$$

It is clear that $K'_+(\bar{Y}) \subseteq K_+(\overline{Y}) \subseteq K_0(\overline{Y})$ and these inclusions are strict in general. All three sets are nonempty (they contain $\bar{Y}$ by definition), star-shaped sets with a center at $\bar{Y}$ which means that if Y belongs to them, then so do the flows $\lambda\bar{Y} + (1 - \lambda)Y, t \in [0, 1]$. The next result is an improved version of Lemma 3.2 of [10] for elementary flows given in Definition 4.1. We give a detailed proof for the readers' convenience.

Proposition 4.2. *The following assertions hold:*

(i) *If a flow Y satisfies the constraint* (2.1) *and if $Y - \overline{Y}$ is a sum of elementary flows, then Y is feasible.*
(ii) *If Y is a feasible flow, then $Y - \overline{Y}$ can be decomposed into a sum of elementary flows in such a way that the sum of $\overline{Y}$ with each of the terms of the decomposition belongs to the set $K_+(\overline{Y})$.*

Proof. To prove the first assertion, it suffices to observe that the sum of a feasible flow with an elementary flow satisfies the constraint (2.2) because for every $i = 1, ..., q$ and $w \in W$ one has $\sum_{j \in J(w)} v_{ji} = v_{\alpha i} - v_{\beta i} = 0$ when

$V = (v_{ji})_{j=1,...,m;i=1,...,q}$ is elementary.
We pass to prove the second assertion. Let us fix a product $i \in \{1, ..., q\}$ and an origin destination pair $w \in W$. Denote by $Y^i(w)$ the portion of the vector Y^i that consists of the components y_{ji} with $j \in J(w)$. We wish to prove the existence of elementary flows $V_t, t = 1, ..., s$ such that

$$
\begin{aligned}
&(4.1) \qquad & Y^i(w) - \overline{Y}^i(w) &= \sum_{t=1}^{s} V_t^i(w) \\
&(4.2) \qquad & V_t^{i'}(w') &= 0 \text{ for all } (i', w') \in \{1, ..., q\} \times W \setminus \{(i, w)\} \\
&(4.3) \qquad & \overline{Y} + V_t &\in K_+(\overline{Y}).
\end{aligned}
$$

Let a denote the vector on the left hand side of (4.1), a^+ its positive part (the negative components of a are set to be zero), and $a^- = a^+ - a$ (the positive components of a are set to be zero and the negative components are set to be their absolute values). Let $\pi(a)$ denote the number of nonzero components of a. We prove the existence of V_t by induction on $\pi(a)$ which depends on a feasible flow Y. It is clear that when $\pi(a) = 0$, the single-product flows Y^i and $\overline{Y}^i$ coincide on the paths joining w, and so the null flow V will satisfy (4.1), (4.2) and (4.3). Assuming that for any feasible flow Y with $\pi(a) \leqq k < |J(w)|$, the existence of elementary flows V_t as above is assured, we now prove it for Y with $\pi(a) = k + 1$. Among components of a^+ and a^- choose a smallest nonzero one, say $a_\alpha = (a^+)_\alpha > 0$ (the case $a_\alpha = (a^-)_\alpha$ is treated in a similar way). Since

$$\sum_{j \in J(w)} a_j = \sum_{j \in J(w)} (Y^i(w) - \overline{Y}^i(w))_j = \sum_{j \in J(w)} (y_{ji} - \overline{y}_{ji}) = d_{wi} - d_{wi} = 0$$

there is some index $\beta \in J(w)$ such that

$$a_\beta = -(a^-)_\beta \leqq -a_\alpha.$$

Define V_0 to be a flow with

$$(V_0)_{ji'} = \begin{cases} a_\alpha & \text{if} \quad i' = i, j = \alpha \\ -a_\alpha & \text{if} \quad i' = i, j = \beta \\ 0 & \text{if} \quad i' = i, j \neq \alpha, \beta, \text{ or } i' \neq i. \end{cases}$$

Then V_0 is an elementary flow. Consider the flow $\hat{Y} = Y - V_0$ and $\hat{a} = \hat{Y}^i(w) - \overline{Y}^i(w)$. We show that it is feasible and has $\pi(\hat{a}) \leqq k$. Indeed, by construction, for every product $i' \neq i$, the flow $\hat{Y}^{i'}$ and $Y^{i'}$ coincide, and the same is true for the product i on the paths connecting $w' \in W, w' \neq w$. As for the product i and the origin destination w, we have

$$\begin{aligned}\sum_{j\in J(w)} \hat{y}_{ji} &= \sum_{j\in J(w)} (Y - V_0)_{ji} \\ &= \sum_{j\in J(w)} y_{ji} - \sum_{j\in J(w)} (V_0)_{ji} \\ &= d_{wi} - ((V_0)_{\alpha i} + (V_0)_{\beta i}) \\ &= d_{wi},\end{aligned}$$

which shows that $\hat{Y}$ satisfies (2.2). Moreover,

$$\hat{y}_{ji} = \begin{cases} y_{ji} & \text{for} \quad j \neq \alpha, \beta \\ y_{\alpha i} - a_\alpha & \text{for} \quad j = \alpha \\ y_{\beta i} + a_\alpha & \text{for} \quad j = \beta \end{cases}$$

in which

$$\begin{aligned} y_{\alpha i} - a_\alpha &= \overline{y}_{\alpha i} \in [l_{\alpha i}, u_{\alpha i}] \\ l_{\beta i} \leqq y_{\beta i} + a_\alpha &\leqq y_{\beta i} + a_\beta = \overline{y}_{\beta i} \leqq u_{\beta i}. \end{aligned}$$

Thus, $\hat{Y}$ satisfies (2.1) and is feasible. To see $\pi(\hat{a}) \leqq k$ it suffices to observe that

$$(\hat{a})_j = \begin{cases} 0 & \text{for} \quad j = \alpha \\ a_\alpha + a_\beta & \text{for} \quad j = \beta \\ a_j & \text{for} \quad j \in J(w) \setminus \{\alpha, \beta\}. \end{cases}$$

By induction for the feasible flow $\hat{Y}$ there exists a finite number of elementary flows $V_1, ..., V_s$ satisfying (4.1), (4.2) and (4.3), with $\hat{Y}$ instead of Y such that $\overline{Y} + V_t, t = 1, ...s$, are feasible. Then

$$Y - \overline{Y} = \hat{Y} - \overline{Y} + V_0 = V_0 + V_1 + ... + V_s$$

in which $\overline{Y} + V_0 = \hat{Y}$ is feasible and so belongs to $K_+(\overline{Y})$ as requested. To complete the proof of ii), it remains to apply the above procedure to all products and all origin destination pairs of W. $\square$

Note that if an origin destination pair w of the network has no more than two paths, then due to the conservation law (2.2) a flow V taking the zero value on the paths connecting other pairs $w' \neq w$ is elementary if and only if $\overline{Y} + V$ is feasible. Evidently, this is not the case when there are more than two paths connecting w.

5. Vector variational inequalities

An important issue of network equilibrium research is to construct an equivalent variational inequality. This is because the theory of variational inequalities is well developed and there are efficient solving methods that can be applied to compute equilibrium of a network. In networks with single-criterion and single-product there has been established a complete equivalence between a network equilibrium problem and a suitably constructed variational inequality problem (see [11] for details). Results on this direction for multi-product multi-criteria networks are far from being satisfactory. In fact, most of the works on multi-product multi-criteria equilibrium use scalarization to obtain a scalar equilibrium, and then its equivalent scalar variational inequality is constructed via standard approach (see [2,7,13]). An attempt has been done in [12] by directly constructing vector variational inequalities. It produces a sufficient condition for equilibrium, but not necessary, and in general it cannot be necessary as it was already analyzed in [7]. Only in a recent paper [10], a full equivalence between a network equilibrium problem and a vector variational inequality problem has been established, for single-product multi-criteria models. The concept of elementary flows does make it possible. Let us move on construction of vector variational inequalities corresponding to the problems of finding weak, strong and ideal equilibrium defined in Section 3 by using the idea of elementary flows.

5.1. **Weak equilibrium.** We are given a feasible pattern flow $\overline{Y}$. Because the entries of the cost matrix $C(\overline{Y})$ are vectors of the space $\mathbb{R}^\ell$, each vector $(c_{j1}, ..., c_{jq})^T$ is itself a real $q \times \ell$-matrix and considered as an element of the space $\mathbb{R}^{\ell\times q}$. We introduce the following variational inequality over elementary flows.

(V1) Find a feasible path flow $\overline{Y}$ such that

$$\begin{pmatrix} \sum_{j=1}^m c_{j1}^1(y_{j1}-\overline{y}_{j1}) & \dots & \sum_{j=1}^m c_{jq}^1(y_{jq}-\overline{y}_{jq}) \\ \dots & \dots & \dots \\ \sum_{j=1}^m c_{j1}^\ell(y_{j1}-\overline{y}_{j1}) & \dots & \sum_{j=1}^m c_{jq}^\ell(y_{jq}-\overline{y}_{jq}) \end{pmatrix} \notin -\mathbb{R}_+^{\ell\times q}\setminus\{0\}$$

for all $Y \in K'_+(\overline{Y})$.

The next result shows that the variational inequality problem (V1) is equivalent to the problem of finding a weak equilibrium.

Theorem 5.1. *A feasible flow $\overline{Y}$ is a weak equilibrium of the network G if and only if it is a solution of the variational inequality problem (V1).*

Proof. Assume that $\overline{Y}$ is a weak vector equilibrium, Y is a feasible flow from $K'_+(\overline{Y})$ and $V := Y - \overline{Y}$. If V is a null flow, we are done because

$$\sum_{j=1}^m c_{ji}(y_{ji}-\overline{y}_{ji}) = 0 \text{ for all } i = 1, ..., q.$$

If V is not null, there are two paths p_α and p_β for an origin destination pair $w \in W$ such that $V_\beta = -V_\alpha > 0$ and the other $V_j, j \neq \alpha, \beta$, are all null. As $\overline{Y}+V$ is feasible, we deduce that

$$\begin{aligned} \overline{Y}_\alpha &= Y_\alpha - V_\alpha > L_\alpha, \\ \overline{Y}_\beta &= Y_\beta - V_\beta < U_\beta. \end{aligned}$$

Since $\overline{Y}$ is a weak equilibrium, it follows that $C_\alpha - C_\beta \ngeq 0$ by which either $C_\alpha - C_\beta = 0$ or there are some indices $i \in \{1, ..., q\}$ and $s \in \{1, ..., \ell\}$ such that

$c^s_{\alpha i} - c^s_{\beta i} < 0$. In the first case, one has

$$\sum_{j=1}^{m} c_{ji}(y_{ji} - \overline{y}_{ji}) = c_{\alpha i}v_{\alpha i} + c_{\beta i}v_{\beta i} = -(c_{\alpha i} - c_{\beta i})v_{\beta i} = 0 \text{ for all } i = 1, ..., q,$$

where $v_{\beta i}$ is the ith component of V_β. This shows that $\overline{Y}$ satisfies (V1). In the second case, we have

$$\sum_{j=1}^{m} c^s_{ji}(y_{ji} - \overline{y}_{ji}) = c^s_{\alpha i}v_{\alpha i} + c^s_{\beta i}v_{\beta i} = -(c^s_{\alpha i} - c^s_{\beta i})v_{\beta i} > 0,$$

which shows that $\overline{Y}$ satisfies (V1) too.
Conversely, if $\overline{Y}$ is not a weak equilibrium, there are some paths p_α and p_β for an origin destination pair $w \in W$ such that

$$C_\alpha - C_\beta \geq 0,\ \overline{Y}_\alpha > L_\alpha \text{ and } \overline{Y}_\beta < U_\beta. \tag{5.1}$$

Choose a strictly positive vector $v \in \mathbb{R}^q$ such that $\overline{Y}_\alpha - v \geqq L_\alpha$ and $\overline{Y}_\beta + v \leqq U_\beta$, and construct a flow V with $V_\alpha = -v, V_\beta = v$ and $V_j = 0$ on the remaining paths of the network. It is clear that $Y := \overline{Y} + V$ is a feasible flow (see also Proposition 4.2) and belongs to $K'_+(\overline{Y})$. Moreover, in view of (5.1) we have $c_{\alpha i} - c_{\beta i} \geqq 0$ for all $i \in \{1, ..., q\}$ and $c_{\alpha i_0} - c_{\beta i_0} \geq 0$ for some i_0. Consequently,

$$\begin{aligned}\sum_{j=1}^{m} c_{ji}(y_{ji} - \overline{y}_{ji}) &= \sum_{j=1}^{m} c_{ji}v_{ji} \\ &= (c_{\beta i} - c_{\alpha i})v_{\beta i} \in -\mathbb{R}^\ell_+\end{aligned}$$

for all $i = 1, ..., q$ and the sum is not equal to 0 for $i = i_0$. Hence $\overline{Y}$ is not a solution of (V1). □

As a consequence of Theorem 5.1, we obtain Theorem 2.1 of [7].

Corollary 5.2. *If a feasible flow $\overline{Y}$ is a solution of the vector variational inequality problems (VI$_i$):*

$$\sum_{j=1}^{m} c_{ji}(y_{ji} - \overline{y}_{ji}) \notin -\mathbb{R}^\ell_+ \setminus \{0\},\ \ Y \in K(\overline{Y}),$$

$i = 1, ..., q$, then it is a vector equilibrium in the sense of (3.2).

Proof. Since $K'_+(\overline{Y})$ is a subset of $K(\overline{Y})$ every solution of the variational inequality of the corollary with i fixed solves the problems (VI'$_i$):

$$\sum_{j=1}^{m} c_{ji}(y_{ji} - \overline{y}_{ji}) \notin -\mathbb{R}^\ell_+ \setminus \{0\},\ \ Y \in K'_+(\overline{Y}).$$

Applying Theorem 5.1 to each single-product network flow $\overline{Y}^i$, we deduce that $\overline{Y}^i$ is a weak equilibrium of the multi-criteria network G with single-product i. This is true for all $i = 1, ..., q$, we conclude that (3.2) is satisfied. □

Since the feasible set $K(\overline{Y})$ is generally larger than the set $K'_+(\overline{Y})$, a weak equilibrium flow is not necessarily a solution of the variational inequality problem of the preceding corollary even in a single-product network. This is shown by the next example.

Example 5.3. Consider a network problem with only one pair origin destination nodes $w = (x, x')$ and only one product to traverse in the network where three paths are available: $P_w = \{p_1, p_2, p_3\}$. Assume:

$$\begin{gathered}\overline{y}_{11} = 2, \quad \overline{y}_{21} = 1, \quad \overline{y}_{31} = 3, \quad d_{w1}(\overline{Y}) = 6, \\ u_{11} = u_{21} = u_{31} = 3, \quad l_{11} = l_{21} = l_{31} = 1, \\ c_{11} = (3,1), \quad c_{21} = (2,2), \quad c_{31} = (1,4).\end{gathered}$$

By choosing the feasible flow Y with components $y_{11} = 1,\ y_{21} = 3,\ y_{31} = 2$, one has

$$\sum_{j=1}^{3} c_{j1}(y_{j1} - \overline{y}_{ji}) = (0, -1) \in -\mathbb{R}^\ell_+ \setminus \{0\},$$

which shows that the flow $\overline{Y}$ does not solve the vector variational problem of Corollary 5.2. So at this stage, it is impossible to say whether $\overline{Y}$ is a vector equilibrium or not (in the sense of (3.2)). Let us now appeal to Theorem 5.1 for help. The set $K'_+(\overline{Y})$ is given by

$$\begin{aligned} K'_+(\overline{Y}) = \overline{Y} + \left\{ \begin{pmatrix} 0 \\ t \\ -t \end{pmatrix} : 0 \le t \le 2 \right\} &\cup \left\{ \begin{pmatrix} t \\ 0 \\ -t \end{pmatrix} : 0 \le t \le 1 \right\} \\ &\cup \left\{ \begin{pmatrix} t \\ -t \\ 0 \end{pmatrix} : -1 \le t \le 0 \right\}. \end{aligned}$$

Then for $Y \in K'_+(\overline{Y})$, we have

$$\begin{aligned} \sum_{j1}^{3} c_{j1}(y_{j1} - \overline{y}_{j1}) &= \begin{cases} (t, -2t)^T & \text{if} \quad Y - \overline{Y} = (0, t, -t)^T, t \in [0, 2] \\ (2t, -3t)^T & \text{if} \quad Y - \overline{Y} = (t, 0, -t)^T, t \in [0, 1] \\ (t, -t)^T & \text{if} \quad Y - \overline{Y} = (t, -t, 0)^T, t \in [-1, 0] \end{cases} \\ &\notin \ -\mathbb{R}^2_+ \setminus \{0\}. \end{aligned}$$

Hence $\overline{Y}$ solves (V1), and by Theorem 5.1, it is a weak vector equilibrium which satisfies (3.2), too, because the network is single-product.

5.2. **Strong equilibrium.** To characterize strong equilibrium, we consider the following vector variational inequality.

(V2) Find a feasible path flow $\overline{Y}$ such that

$$\begin{pmatrix} \sum_{j=1}^m c^1_{j1}(Y_j - \overline{Y}_j) & \dots & \sum_{j=1}^m c^1_{jq}(Y_j - \overline{Y}_j) \\ \dots & \dots & \dots \\ \sum_{j=1}^m c^\ell_{j1}(Y_j - \overline{Y}_j) & \dots & \sum_{j=1}^m c^\ell_{jq}(Y_j - \overline{Y}_j) \end{pmatrix} \notin (-\mathbb{R}^q_+)^{\ell \times q} \setminus \{0\}$$

for all $Y \in K_+(\overline{Y})$.

Given a path flow Y on G, we shall make use of the following condition: for every origin destination pair $w \in W$, for every couple of paths $p_\alpha, p_\beta \in P_w$ one has implication

(5.2) $L_\alpha \leq Y_\alpha$ and $Y_\beta \leq U_\beta \Rightarrow \; l_{\alpha k} < y_{\alpha k}$ and $y_{\beta k} < u_\beta$ for some $k \in \{1, ..., q\}$.

We notice that this condition is always satisfied in the case of flows without capacity constraints. In a network with capacity constrains, it can be satisfied if the upper bound on each path is sufficiently large with regard to the demand. For instance, it is the case when $d_{wi} < u_{ji}$ for every $j \in J(w)$, $i \in \{1, ..., q\}$.

Theorem 5.4. *Every strong equilibrium flow* $\overline{Y}$ *is a solution of* (V2)*. Moreover if a solution of* (V2) *satisfies* (5.2)*, then it is a strong equilibrium. In particular, when the network is without capacity constraints, a feasible flow is a strong equilibrium if and only if it is a solution of* (V2)*.*

Proof. Assume that $\overline{Y}$ is a strong equilibrium, Y is a feasible flow from $K_+(\overline{Y})$ and $V := Y - \overline{Y}$. If V is a null flow, (V2) holds evidently. If V is not null, there are two paths p_α and p_β for an origin destination pair $w \in W$ such that $V_\beta = -V_\alpha \geq 0$ and the other $V_j, j \neq \alpha, \beta$, are all null. As $\overline{Y} + V$ is feasible, we deduce that

$$\overline{Y}_\alpha \geq L_\alpha \text{ and } \overline{Y}_\beta \leq U_\beta.$$

Since $\overline{Y}$ is a strong equilibrium, it follows that $C_\alpha - C_\beta \not\geq 0$ by which either $C_\alpha - C_\beta = 0$ or there are some indices i_* and s_* such that $c^{s_*}_{\alpha i_*} - c^{s_*}_{\beta i_*} < 0$. In the first case, we have

$$\sum_{j=1}^{m} c^s_{ji}(Y_j - \overline{Y}_j) = c^s_{\alpha i}V_\alpha + c^s_{\beta i}V_\beta = -(c^s_{\alpha i} - c^s_{\beta i})V_\beta = 0 \text{ for all } i = 1, ..., q, s = 1, ..., \ell,$$

and in the second case

$$\sum_{j=1}^{m} c^{s_*}_{ji_*}(Y_j - \overline{Y}_j) = -(c^{s_*}_{\alpha i_*} - c^{s_*}_{\beta i_*})V_\beta \geq 0,$$

which shows that $\overline{Y}$ is a solution of (V2).
Conversely, let $\overline{Y}$ be a solution of (V2) that satisfies (5.2). If it is not an equilibrium, there are some paths p_α and p_β for an origin destination pair $w \in W$ such that

$$C_\alpha - C_\beta \geq 0, \; \overline{Y}_\alpha \geq L_\alpha \text{ and } \overline{Y}_\beta \leq U_\beta.$$

We have $c_{\alpha i} - c_{\beta i} \geqq 0$ for all i and $c_{\alpha i_0} - c_{\beta i_0} \geq 0$ for some i_0. Moreover, in view of (5.2), there is some index $k \in \{1, ..., q\}$ such that $\overline{y}_{\alpha k} > l_{\alpha k}$ and $\overline{y}_{\beta k} < u_{\beta k}$. Choose a small number ε such that $\overline{y}_{\alpha k} - \varepsilon \geqq l_{\alpha k}$ and $\overline{y}_{\beta k} + \varepsilon \leqq u_{\beta k}$, and define V to be an elementary flow with $v_{\alpha k} = -\varepsilon$, $v_{\beta k} = \varepsilon$ and the other components equal to zero. Then $Y := \overline{Y} + V$ belongs to $K_+(\overline{Y})$ for which

$$\begin{aligned}\sum_{j=1}^{m} c^s_{ji}(Y_j - \overline{Y}_j) &= \sum_{j=1}^{m} c^s_{ji} V_j \\ &= (c^s_{\beta i} - c^s_{\alpha i})V_\beta \leqq 0\end{aligned}$$

for all $i = 1, ..., q, s = 1, ..., \ell$ and the sum is not zero for $i = i_0, s = s_0$. This contradicts the hypothesis. □

We note that problems (V1) and (V2) are the same when the network is single-product because the sets of elementary flows $K_+(\overline{Y})$ and $K'_+(\overline{Y})$ coincide in such a model.

5.3. **Ideal equilibrium.** A more familiar formulation of variational inequalities is obtained when the sign of "does not belong to" in (V2) is replaced by inequalities. Such a formulation is given next.

(V3) Find a feasible path flow $\overline{Y}$ such that

$$\begin{pmatrix} \sum_{j=1}^m c^1_{j1}(Y_j - \overline{Y}_j) & ... & \sum_{j=1}^m c^1_{jq}(Y_j - \overline{Y}_j) \\ ... & ... & ... \\ \sum_{j=1}^m c^\ell_{j1}(Y_j - \overline{Y}_j) & ... & \sum_{j=1}^m c^\ell_{jq}(Y_j - \overline{Y}_j) \end{pmatrix} \in (\mathbb{R}^q_+)^{\ell \times q}$$

for all $Y \in K_+(\overline{Y})$.

Note that the positive orthant $(\mathbb{R}^q_+)^{\ell\times q}$ is a part of the complement of the negative orthant $-(\mathbb{R}^q_+)^{\ell\times q}\backslash\{0\}$, therefore solutions of (V3) are also solutions of (V2). The converse is evidently not true in general.

Theorem 5.5. *If a feasible flow $\overline{Y}$ is an ideal equilibrium, then it is a solution of* (V3). *Conversely, if the network is without capacity constraints and if a feasible flow $\overline{Y}$ is a solution of* (V3), *then it is an ideal equilibrium.*

Proof. Assume that $\overline{Y}$ is an ideal equilibrium and Y is a feasible flow from $K_+(\overline{Y})$. Let $V := Y - \overline{Y}$. There are two paths p_α and p_β for an origin destination pair $w \in W$ such that $V_\beta = -V_\alpha \geqq 0$ and the other $V_j, j \neq \alpha, \beta$, are all null. We have (5.3)

$$\sum_{j=1}^{m} c^s_{ji}(Y_j - \overline{Y}_j) = (c^s_{\alpha i} - c^s_{\beta i})V_\alpha = (c^s_{\beta i} - c^s_{\alpha i})V_\beta, \text{ for all } i = 1, ..., k; s = 1, ..., \ell.$$

If $C_\alpha \geq \mathrm{Inf}\,(C(w))$, then $\overline{Y}_\alpha = L_\alpha$ and $V_\alpha = 0$, implying that the expression in the middle of (5.3) is equal to zero. If $C_\alpha = \mathrm{Inf}\,(C(w))$, then $C_\beta \geqq C_\alpha$ and the expression on the right hand side of (5.3) is positive. By this, $\overline{Y}$ is a solution of (V3).

Conversely, let $\overline{Y}$ be a feasible flow which is a solution of (V3). If it is not an ideal equilibrium, one may find some $w \in W$, $p_\alpha, p_\beta \in P_w$, $s \in \{1, ..., \ell\}$ and $i, k \in \{1, ..., q\}$ such that $c^s_{\alpha i} > c^s_{\beta i}$, which means that $C_\alpha \neq \mathrm{Inf}\,(C(w))$, and $\overline{y}_{\alpha k} > 0$. Define Y to be a flow with $y_{\alpha k} = 0, y_{\beta k} = \overline{y}_{\alpha k} + \overline{y}_{\beta k}$ and $y_{ji} = \overline{y}_{ji}$ for other j and i. It is evident that Y is feasible and belongs to $K_+(\overline{Y})$. Moreover, one has

$$\begin{aligned}\sum_{j=1}^{m} c^s_{ji}(Y_j - \overline{Y}_j) &= c^s_{\alpha i}(Y_\alpha - \overline{Y}_\alpha) + c^s_{\beta i}(Y_\beta - \overline{Y}_\beta)\\ &= \overline{Y}_\alpha(c^s_{\beta i} - c^s_{\alpha i}) \leq 0,\end{aligned}$$

which contradicts the hypothesis. □

A particular case of Theorem 5.5 has been treated by Cheng and Wu (Section 2, [2]) for a single-criterion multi-product model without capacity constraints. Actually the above authors consider the variational inequality problem (with $\ell = 1$):

(V4) find a feasible path flow $\overline{Y}$ such that

$$\left(\sum_{j=1}^{m} c^1_{j1}(Y_j - \overline{Y}_j), ..., \sum_{j=1}^{m} c^1_{jq}(Y_j - \overline{Y}_j)\right) \in (\mathbb{R}^q_+)^q \text{ for all } Y \in K(\overline{Y}),$$

and prove (Theorem 2.1, [2]) that a feasible flow $\overline{Y}$ is an ideal equilibrium (called a Wardrop equilibrium in the sense of (H1) of Section 2) if and only if it is a solution to (V4). By the definition, the set of feasible flows $K(\overline{Y})$ contains the set $K_+(\overline{Y})$. Therefore, every solution of (V4) is a solution of (V3). It turns out that the converse is also true which shows that (V3) and (V4) are equivalent. Indeed, given a feasible flow Y, applying Proposition 4.2, we obtain a decomposition of $Y - \overline{Y}$ by a sum $V_1 + ... + V_s$ with $\overline{Y} + V_t \in K_+(\overline{Y})$, $t = 1, ..., s$. If $\overline{Y}$ solves (V3), then all vectors on the left hand side of (V3) with $(V_t)_j, t = 1, ..., s$ instead of $(Y_j - \overline{Y}_j)$ are positive, hence their sum over i is positive, which proves that $\overline{Y}$ solves (V4).

We close up this section by a remark that the second part of Theorem 5.5 can be formulated for flows with capacity constraints under certain hypotheses similar to that given in (5.2). We do not go into details of such hypotheses because as already said the main concern of multi-product networks is not ideal equilibrium, but its weaker version such as weak equilibrium or strong equilibrium stated in Theorems 5.1 and 5.4.

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Laboratoire de Mathématiques EA 2151
University of Avignon, France;
and
Hanoi Institute of Mathematics, Vietnam
E-mail address: dtluc@univ-avignon.fr