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MINIMAX VARIATIONAL INEQUALITIES

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Dedicated to Tran Duc Van on the occasion of his sixtieth birthday

ABSTRACT. We introduce a new notion called *minimax variational inequality* (MVI). The solution existence of nonmonotone MVIs in Euclidean spaces, pseudomonotone MVIs in reflexive Banach spaces, and strongly monotone MVIs in Hilbert spaces is studied in detail. We show that MVIs can serve as a good tool for studying minimax problems given by convex sets and differentiable functions.

1. Introduction to MVIs

Minimax variational inequality (MVI for short) is a new mathematical model which is considered systematically for the first time, as far as we understand, in this paper. Naturally, one may pose the following questions:
(a) What is MVI?
(b) Why to study MVIs?
(c) What is the difference between the new model and the celebrated notion of *variational inequality* (VI for short)?

Questions (a) and (c) will be answered later, in the final part of this section. Question (b) can be answered simply by saying that *MVIs deserve a study because they provide us with a good tool for studying minimax problems given by convex sets and differentiable functions.* The role of MVIs for differentiable minimax problems is quite similar to that of VIs for differentiable optimization problems. We refer to [10, 12, 24] for an explanation about the two-way connection between VIs and differentiable optimization problems with convex constraint sets. Before going further, we need to recall some basic facts about minimax problems and saddle points.

Let K, L be nonempty closed convex sets in Banach spaces X and Y, respectively. Suppose that $f : \Omega \to \mathbb{R}$ is a Fréchet continuously differentiable function

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defined on an open subset Ω of $X \times Y$ with $K \times L \subset \Omega$. The *minimax problem* given by the convex sets K, L and the function f, which is written formally as

$$\max_{y\in L} \min_{x\in K} f(x,y), \tag{1.1}$$

is that one of finding a point $(\bar{x},\bar{y}) \in K \times L$ such that

$$f(\bar{x},y) \le f(\bar{x},\bar{y}) \le f(x,\bar{y}) \quad \forall x \in K,\ \forall y \in L. \tag{1.2}$$

If $(\bar{x},\bar{y}) \in K \times L$ satisfies (1.2), then one says that it is a *saddle point* of the minimax problem (1.1). Following [16] we put

$$\eta = \sup_{y\in L} \inf_{x\in K} f(x,y), \quad \gamma = \inf_{x\in K} \sup_{y\in L} f(x,y).$$

From the formulae it follows that $\eta \le \gamma$. If $(\bar{x},\bar{y})$ is a saddle point of (1.1), then it is easy to show that $\eta \ge f(\bar{x},\bar{y}) \ge \gamma$; hence $\eta = \gamma = f(\bar{x},\bar{y})$. Therefore, the existence of saddle points implies that

$$\sup_{y\in L} \inf_{x\in K} f(x,y) = \inf_{x\in K} \sup_{y\in L} f(x,y).$$

The common value $\eta = \gamma = f(\bar{x},\bar{y})$ is called the *saddle value* of (1.1). Note that the equality $\eta = \gamma$ may hold even in the case there are no saddle points.

To make the presentation more pleasant for reading, proofs will be provided for the following standard *necessary* and *sufficient* conditions for saddle points.

Theorem 1.1. *If* $(\bar{x},\bar{y}) \in K \times L$ *is a saddle point of* (1.1), *then*

$$\langle F_2(\bar{x},\bar{y}), y-\bar{y}\rangle \le 0 \le \langle F_1(\bar{x},\bar{y}), x-\bar{x}\rangle \quad \forall x \in K,\ \forall y \in L, \tag{1.3}$$

where $F_1(u,v) := \nabla_x f(u,v)$ *and* $F_2(u,v) := \nabla_y f(u,v)$ *denote respectively the partial gradients of* $f(x,y)$ *at* (u,v) *with respect to* x *and* y.

Proof. Suppose that $(\bar{x},\bar{y}) \in K \times L$ is a saddle point of (1.1). Let $(x,y) \in K \times L$ be given arbitrarily. Since $y_t := \bar{y} + t(y-\bar{y}) = (1-t)\bar{y} + ty$ belongs to L for every $t \in (0,1)$ and since the first inequality in (1.2) holds for any $y \in L$, we obtain

$$\langle F_2(\bar{x},\bar{y}), y-\bar{y}\rangle = \nabla_y f(\bar{x},\bar{y})(y-\bar{y}) = \lim_{t\downarrow 0} \frac{f(\bar{x},y_t) - f(\bar{x},\bar{y})}{t} \le 0,$$

which establishes the first inequality in (1.3). The second inequality of (1.3) can be proved in the same manner. □

Theorem 1.2. *Suppose that, for every* $(x,y) \in K \times L$, $f(\cdot,y)$ *is pseudo-convex on* K *and* $f(x,\cdot)$ *is pseudo-concave on* L, *i.e.*,

$$\Big(u,u' \in K,\ \langle \nabla_x f(u,y), u'-u\rangle \ge 0\Big) \Rightarrow f(u',y) - f(u,y) \ge 0$$

and

$$\Big(v,v' \in K,\ \langle \nabla_y f(x,v), v'-v\rangle \le 0\Big) \Rightarrow f(x,v') - f(x,v) \le 0.$$

If $(\bar{x},\bar{y}) \in K \times L$ *satisfies condition* (1.3) *where* $F_1(u,v) := \nabla_x f(u,v)$ *and* $F_2(u,v) := \nabla_y f(u,v)$, *then* $(\bar{x},\bar{y}) \in K \times L$ *is a saddle point of* (1.1). *In particular, if* (1.3) *is valid and* $f(\cdot,y)$ *is convex on* K *and* $f(x,\cdot)$ *is concave on* L *for every fixed pair* $(x,y) \in K \times L$, *then* $(\bar{x},\bar{y}) \in K \times L$ *is a saddle point of* (1.1).

Proof. Suppose that $(\bar{x}, \bar{y}) \in K \times L$ and (1.3) holds. Given any $x \in K$, from the second inequality in (1.3) we can deduce that

$$\langle \nabla_x f(\bar{x}, \bar{y}), x - \bar{x} \rangle = \nabla_x f(\bar{x}, \bar{y})(x - \bar{x}) = \langle F_1(\bar{x}, \bar{y}), x - \bar{x} \rangle \geq 0.$$

Combining this with the pseudo-convexity of $f(\cdot, \bar{y})$ yields $f(x, \bar{y}) - f(\bar{x}, \bar{y}) \geq 0$ and establishes the second inequality in (1.2). The first inequality of (1.2) can be proved similarly. □

Theorem 1.1 hints that it is worthy to consider (1.3) as a mathematical model standing independently from the source problem (1.1). To make a good credit to the origin of the model and to stress its potential applications back to the source problem (1.1), we may call it a "minimax variational inequality".

Definition 1.3. Let X, Y be Banach spaces with the dual spaces denoted respectively by X^* and Y^*. Let $K \subset X$, $L \subset Y$ be nonempty closed convex sets, and let $F_1 \colon K \times L \to X^*$, $F_2 \colon K \times L \to Y^*$ be arbitrarily given functions. The *minimax variational inequality* defined by the data set $\{K, L, F_1, F_2\}$ is the problem of finding a point $(\bar{x}, \bar{y}) \in K \times L$ such that

$$(MVI) \qquad \langle F_2(\bar{x}, \bar{y}), y - \bar{y} \rangle \leq 0 \leq \langle F_1(\bar{x}, \bar{y}), x - \bar{x} \rangle \quad \forall x \in K, \ \forall y \in L.$$

The solution set of (MVI) is abbreviated to $\mathrm{Sol}(MVI)$.

Remark 1.4. According to Theorem 1.1, if the solution set of (1.1) is denoted by S then it holds $S \subset \mathrm{Sol}(MVI)$, provided that we put $F_1 = \nabla_x f$ and $F_2 = \nabla_y f$. Moreover, if $f(\cdot, y)$ is pseudo-convex on K and $f(x, \cdot)$ is pseudo-concave on L for every $(x, y) \in K \times L$, then $S = \mathrm{Sol}(MVI)$ by Theorem 1.2. Thus, (MVI) can be used in studying the minimax problem (1.1).

Remark 1.5. The notion of pseudo-convex function plays an important role in optimization theory (see e.g. [11, Chapter 9]). According to [11, Theorem 5, p. 143], if $f(\cdot, y)$ is pseudo-convex on K, then it is *quasiconvex* on K in the sense that

$$f((1-t)x + tu, y) \leq \max\{f(x, y), f(u, y)\} \quad \forall x, u \in K, \ \forall t \in (0, 1).$$

Moreover, this $f(\cdot, y)$ is also *strictly quasiconvex* on K, i.e.,

$$\begin{gathered} f((1-t)x + tu, y) < \max\{f(x, y), f(u, y)\} \\ \forall x, u \in K, \ f(x, y) \neq f(u, y), \ \forall t \in (0, 1). \end{gathered}$$

Quasiconvexity and quasiconcavity of functions are basic assumptions in some minimax theorems (see for instance Sion's theorem [1, Theorem 7, p. 218] and the related deep results in [16, Section 2]). We refer to [3, Theorem 2.1(ii), p. 92] for an interesting sufficient condition for having the implication quasiconvexity $\Rightarrow$ pseudoconvexity. Fundamental facts about generalized convexity of functions and monotonicity of operators can be found in the handbook [7].

Remark 1.6. Problem (1.1) can be interpreted as a two-person zero-sum game (see [1, Chapter 7], [2, p. 312]). Despite its importance in many applications, (1.1) only represents a standard minimax problem. We refer to [1, 2, 15] for the fundamentals of minimax theory and to [13, 16] for some recent results on the stability of saddle points and/or saddle values, and on minimax theorems. New advances in minimax theory with applications to studying the solution multiplicity of nonlinear equations and well-posedness of optimization problems are presented in [14]. Some applications of the lop-sided minimax theorem to differential stability of optimization problems involving set-valued maps and to solution existence of generalized quasi-variational inequalities can be found respectively in [6] and [4].

Consider problem (MVI) and put $G(x,y) = (F_1(x,y), -F_2(x,y))$ for all $(x,y) \in K \times L$. Thus, the value of the functional $G(x,y) \in X^* \times Y^*$ at $(u,v) \in X \times Y$ is given by

$$\langle G(x,y),(u,v)\rangle = \langle F_1(x,y),u\rangle - \langle F_2(x,y),v\rangle. \tag{1.4}$$

Unless otherwise stated, the norm in the product space $X \times Y$ is defined by setting $\|(x,y)\| = \|x\| + \|y\|$. We are interested in the variational inequality defined by the closed convex set $K \times L \subset X \times Y$ and the operator $G : K \times L \to X^* \times Y^*$:

$$\text{Find } (\bar{x},\bar{y}) \in K \times L \text{ s.t. } \langle G(\bar{x},\bar{y}),(x,y)-(\bar{x},\bar{y})\rangle \geq 0 \quad \forall (x,y) \in K \times L. \tag{1.5}$$

Proposition 1.7. *The inclusion* $(\bar{x},\bar{y}) \in \mathrm{Sol}(MVI)$ *holds if and only if* $(\bar{x},\bar{y})$ *is a solution of* (1.5).

Proof. If $(\bar{x},\bar{y}) \in \mathrm{Sol}(MVI)$, then for every $(x,y) \in K \times L$, we have

$$\langle F_2(\bar{x},\bar{y}),y-\bar{y}\rangle \leq 0 \leq \langle F_1(\bar{x},\bar{y}),x-\bar{x}\rangle. \tag{1.6}$$

It is easy to see that (1.6) implies $\langle G(\bar{x},\bar{y}),(x,y)-(\bar{x},\bar{y})\rangle \geq 0$ for every $(x,y) \in K \times L$. Conversely, if $(\bar{x},\bar{y})$ is a solution of (1.5), then

$$\langle G(\bar{x},\bar{y}),(x,y)-(\bar{x},\bar{y})\rangle \geq 0 \quad \forall (x,y) \in K \times L.$$

Taking $x = \bar{x}$, from the last condition we can deduce that $\langle F_2(\bar{x},\bar{y}),y-\bar{y}\rangle \leq 0$ for every $y \in L$. Similarly, by choosing $y = \bar{y}$ we get $0 \leq \langle F_1(\bar{x},\bar{y}),x-\bar{x}\rangle$ for every $x \in K$. This establishes (1.6), which shows that $(\bar{x},\bar{y}) \in \mathrm{Sol}(MVI)$. □

For variational inequalities, coercivity, monotonicity, strict monotonicity, pseudomonotonicity, strict pseudomonotonicity, and strong monotonicity are fundamental concepts; see e.g. [7, 9, 10, 19, 20, 21, 22, 23]. Applied to the map $G = (F_1, -F_2) : K \times L \to X^* \times Y^*$ given in (1.4) and the variational inequality (1.5), coercivity, monotonicity, strict monotonicity, pseudomonotonicity, and strong monotonicity in theory of VIs mean the following:

(i) Problem (1.5) is said to satisfy the *coercivity condition* if there exists a point $(x_0,y_0) \in K \times L$ such that

$$\lim_{\substack{\|(x,y)\|\to\infty \\ (x,y)\in K\times L}} \frac{\langle G(x,y)-G(x_0,y_0),(x,y)-(x_0,y_0)\rangle}{\|x-x_0\|+\|y-y_0\|} = +\infty. \tag{1.7}$$

(ii) Problem (1.5) is said to be *monotone* if

$$\langle G(x,y)-G(u,v),(x-u,y-v)\rangle \geq 0 \quad \forall (x,y),(u,v) \in K \times L.$$

(iii) Problem (1.5) is said to be *strictly monotone* if

$$\langle G(x,y) - G(u,v), (x-u, y-v)\rangle > 0$$
$$\forall (x,y),(u,v) \in K \times L,\ (x,y) \neq (u,v).$$

(iv) Problem (1.5) is said to be *pseudomonotone* if

$$\Big((x,y),(u,v) \in K \times L,\ \langle G(u,v), (x-u, y-v)\rangle \geq 0\Big)$$
$$\Longrightarrow \langle G(x,y), (x-u, y-v)\rangle \geq 0.$$

(v) Problem (1.5) is said to be *strictly pseudomonotone* if

$$\Big((x,y),(u,v) \in K \times L,\ (x,y) \neq (u,v),\ \langle G(u,v), (x-u, y-v)\rangle \geq 0\Big)$$
$$\Longrightarrow \langle G(x,y), (x-u, y-v)\rangle > 0.$$

(vi) Problem (1.5) is said to be *strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle G(x,y) - G(u,v), (x-u, y-v)\rangle \geq \alpha(\|x-u\|^2 + \|y-v\|^2) \quad \forall (x,y),(u,v) \in K \times L. \tag{1.8}$$

The implications strong monotonicity ⇒ strict monotonicity, strict monotonicity ⇒ monotonicity, monotonicity ⇒ pseudomonotonicity, strict monotonicity ⇒ strict pseudomonotonicity, and strong monotonicity ⇒ coercivity, are well known. Remembering that $G = (F_1, -F_2)$, we can rewrite the definitions (i)–(vi) equivalently as follows.

Definition 1.8. (MVI) is said to satisfy the *coercivity condition* if there exists a point $(x_0, y_0) \in K \times L$ such that

$$\lim_{\substack{\|(x,y)\|\to\infty \\ (x,y)\in K\times L}} \frac{\langle F_1(x,y) - F_1(x_0,y_0), x - x_0\rangle - \langle F_2(x,y) - F_2(x_0,y_0), y - y_0\rangle}{\|x-x_0\| + \|y-y_0\|} = +\infty. \tag{1.9}$$

Definition 1.9. (MVI) is said to be a *monotone minimax variational inequality* if

$$\langle F_1(x,y) - F_1(u,v), x-u\rangle - \langle F_2(x,y) - F_2(u,v), y-v\rangle \geq 0 \quad \forall (x,y),(u,v) \in K \times L.$$

Definition 1.10. (MVI) is said to be a *strictly monotone minimax variational inequality* if

$$\langle F_1(x,y) - F_1(u,v), x-u\rangle - \langle F_2(x,y) - F_2(u,v), y-v\rangle > 0 \quad \forall (x,y),(u,v) \in K \times L,\ (x,y) \neq (u,v).$$

Definition 1.11. (MVI) is said to be a *pseudomonotone minimax variational inequality* if

$$\Big((x,y),(u,v) \in K \times L,\ \langle F_1(u,v), x-u\rangle - \langle F_2(u,v), y-v\rangle \geq 0\Big)$$
$$\Longrightarrow \langle F_1(x,y), x-u\rangle - \langle F_2(x,y), y-v\rangle \geq 0.$$

Definition 1.12. (MVI) is said to be a *strictly pseudomonotone minimax variational inequality* if

(1.10)
$$\big((x,y),(u,v)\in K\times L,\ (x,y)\neq(u,v),\ \langle F_1(u,v),x-u\rangle-\langle F_2(u,v),y-v\rangle\geq 0\big)$$
$$\Longrightarrow \langle F_1(x,y),x-u\rangle-\langle F_2(x,y),y-v\rangle>0.$$

Definition 1.13. (MVI) is said to be a *strongly monotone minimax variational inequality* if there exists a constant $\alpha>0$ such that

$$\begin{aligned}&\langle F_1(x,y)-F_1(u,v),x-u\rangle-\langle F_2(x,y)-F_2(u,v),y-v\rangle\\ &\geq\alpha(\|x-u\|^2+\|y-v\|^2)\quad\forall(x,y),(u,v)\in K\times L.\end{aligned}\tag{1.11}$$

Remark 1.14. Strictly pseudomonotone MVI can have at most one solution. This fact follows from the above definitions, Proposition 1.7, and [19, Lemma 3.2].

Let us consider a mimimax problem which leads to a monotone MVI.

Example 1.15. Consider problem (1.1) where $X=\mathbb{R}^n$, $Y=\mathbb{R}^m$, $f(x,y)=x^TBy$, with B being an $n\times m$ matrix and ^{T} denoting the matrix transposition. Since $F_1(x,y)=\nabla_x f(x,y)=By$, $F_2(x,y)=\nabla_y f(x,y)=B^Tx$, we have $G(x,y)=(By,-B^Tx)$. Therefore

$$\begin{aligned}&\langle G(x,y)-G(u,v),(x-u,y-v)\rangle\\ &=\langle By-Bv,x-u\rangle+\langle -B^Tx+B^Tu,y-v\rangle\\ &=\langle B(y-v),x-u\rangle-\langle x-u,B(y-v)\rangle=0.\end{aligned}$$

This means that the MVI corresponding to the minimax problem under consideration is monotone.

We now look at another, more general, mimimax problem which leads to strongly monotone MVIs.

Example 1.16. Consider problem (1.1) where $X=\mathbb{R}^n$, $Y=\mathbb{R}^m$,

$$f(x,y)=\frac{1}{2}x^TAx+x^TBy-\frac{1}{2}y^TCy+a^Tx+b^Ty$$

with $A\in\mathbb{R}^{n\times n}$ and $C\in\mathbb{R}^{m\times m}$ being symmetric matrices, $B\in\mathbb{R}^{n\times m}$, $a\in\mathbb{R}^n$, and $b\in\mathbb{R}^m$. Since

$$F_1(x,y)=\nabla_x f(x,y)=Ax+By+a,\quad F_2(x,y)=\nabla_y f(x,y)=B^Tx-Cy+b,$$

it holds

$$G(x,y)=(Ax+By+a,-B^Tx+Cy-b).$$

Hence

$$\begin{aligned}&\langle G(x,y)-G(u,v),(x-u,y-v)\rangle\\ &=\langle A(x-u)+B(y-v),x-u\rangle+\langle -B^T(x-u)+C(y-v),y-v\rangle\\ &=\langle A(x-u),x-u\rangle+\langle C(y-v),y-v\rangle+\langle B(y-v),x-u\rangle-\langle x-u,B(y-v)\rangle\\ &=\langle A(x-u),x-u\rangle+\langle C(y-v),y-v\rangle.\end{aligned}$$

From the latter it follows that *the MVI corresponding to the minimax problem under consideration is strongly monotone if and only if* $u^TAu>0$ *for all* $u\in(\operatorname{span}K)\setminus\{0\}$ *and* $v^TCv>0$ *for all* $v\in(\operatorname{span}L)\setminus\{0\}$, *with* $\operatorname{span}M$ *denoting* *the linear subspace generated by M.* It is also clear that *the MVI in question is monotone if and only if $u^T Au \geq 0$ for all $u \in \operatorname{span} K$ and $v^T Cv \geq 0$ for all $v \in \operatorname{span} L$.* Interestingly, for this MVI, the strict monotonicity is equivalent to the strong monotonicity. Observe also that from the well known second-order characterization of the convexity of differentiable real-valued functions [17] (see also [11, 15, 18]) it follows that the property "$u^T Au \geq 0$ for all $u \in \operatorname{span} K$ and $v^T Cv \geq 0$ for all $v \in \operatorname{span} L$" is equivalent to the convexity of $f(\cdot, y)$ for all $y \in L$ and the concavity of $f(x, \cdot)$ for all $x \in K$. Thus, the monotonicity of this MVI is equivalent to the requirement that $f(x, y)$ is a convex-concave function on $K \times L$.

Now we can answer question (a) stated at the beginning of this section by saying that MVI is an analogue of the well-known variational inequality model, which gives us a convenient tool for dealing with minimax problems of the form (1.1).

To answer question (c), we observe that although any MVI can be transformed to a VI by the trick described in Proposition 1.7 but, as shown in Definitions 1.8–1.13, the operator F_1 (resp., F_2) is attached to variable x (resp., y) tighter than to the second variable (resp., the first variable). In result, one has a VI with a decomposable structure.

It is worthy observing that if L is a singleton, then (MVI) collapses to the classical VI. Thus, (MVI) is an extension of the latter.

In this paper, we focus on the solution existence and solution uniqueness of MVIs. The solution stability and sensitivity of MVIs are considered in [8]. Solution methods will be discussed in a subsequent paper.

The remainder of this paper has 4 sections. Solution existence for nonmonotone MVIs in Euclidean spaces is discussed in Section 2. Section 3 investigates the solution existence of pseudomonotone MVIs in reflexive Banach spaces. Section 4 establishes the solution existence and uniqueness of strongly monotone MVIs in Hilbert spaces. Applications of the obtained results to minimax problems are derived in each of these three sections. Several useful examples are given in the last section.

2. Nonmonotone MVIs in Euclidean Spaces

In this section, it is assumed that $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$. Then X^* and Y^* can be identified with $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. The value of $x^* \in X^*$ at $x \in X$ is identified with the inner product $\langle x^*, x\rangle$ of two vectors in $\mathbb{R}^n$.

The Hartman-Stampacchia theorem [10] can be applied only to variational inequalities in finite-dimensional Euclidean spaces. Its key feature is that it does not require any monotonicity assumption.

Theorem 2.1. (see [10, Theorem 3.1, p. 12]) *If $K \subset \mathbb{R}^n$ is a nonempty compact convex subset and $F : K \to \mathbb{R}^n$ is a continuous function, then the variational inequality problem*

$$\text{Find } \bar{x} \in K \text{ s.t. } \langle F(\bar{x}), x - \bar{x}\rangle \geq 0 \quad \forall x \in K$$

has a solution.

We now derive an analogue of the Hartman-Stampacchia theorem for MVIs.

Theorem 2.2. *Suppose that $K \subset \mathbb{R}^n$, $L \subset \mathbb{R}^m$ are nonempty compact convex subsets, and $F_1 : K \times L \to \mathbb{R}^n$, $F_2 : K \times L \to \mathbb{R}^m$ are continuous functions. Then the minimax variational inequality problem (MVI) has a solution.*

Proof. Let $G : K \times L \to \mathbb{R}^n \times \mathbb{R}^m$ be defined by formula (1.4). From our assumptions it follows that $K \times L \subset \mathbb{R}^n \times \mathbb{R}^m$ is a nonempty compact convex set, G is a continuous map. According to Theorem 2.1, (1.5) has a solution $(\bar{x}, \bar{y}) \in K \times L$. Invoking Proposition 1.7 we can conclude that $(\bar{x}, \bar{y}) \in \mathrm{Sol}(MVI)$. □

Remark 2.3. If L is a singleton, then Theorem 2.2 recovers Theorem 2.1.

Theorem 2.2 yields the next statement on the existence of saddle points. This result is not new. In fact, it is a corollary of Sion's minimax theorem (see e.g. [1, Theorem 7, p. 218]).

Theorem 2.4. *Consider the minimax problem* (1.1) *and assume that $K \subset \mathbb{R}^n$ and $L \subset \mathbb{R}^m$ are nonempty compact convex subsets. If $f(\cdot, y)$ is pseudo-convex on K and $f(x, \cdot)$ is pseudo-concave on L for every $(x, y) \in K \times L$, then* (1.1) *has a saddle point. In particular, if $f(\cdot, y)$ is convex on K and $f(x, \cdot)$ is concave on L for every fixed pair $(x, y) \in K \times L$, then* (1.1) *has a saddle point.*

Proof. Put $F_1(u, v) = \nabla_x f(u, v)$ and $F_2(u, v) = \nabla_y f(u, v)$ for every $(u, v) \in K \times L$. By Theorem 2.2, (MVI) has a solution $(\bar{x}, \bar{y}) \in K \times L$. The assertions now follow from applying Theorem 1.2. □

If K and F are unbounded, then a coercivity condition likes the one in Definition 1.8 must be imposed. Otherwise, the minimax variational inequality under consideration may have no solutions.

Theorem 2.5. *Suppose that $K \subset \mathbb{R}^n$, $L \subset \mathbb{R}^m$ are nonempty closed convex subsets, and $F_1 : K \times L \to \mathbb{R}^n$, $F_2 : K \times L \to \mathbb{R}^m$ are continuous functions. If (MVI) satisfies the coercivity condition, then it has a solution.*

Proof. Our assumptions imply that the map $G = (F_1, -F_2) : K \times L \to \mathbb{R}^n \times \mathbb{R}^m$ is continuous and the coercivity condition (1.7) is satisfied for some point $(x_0, y_0) \in K \times L$. By [10, Corollary 4.3, p. 14], there is a solution $(\bar{x}, \bar{y}) \in K \times L$ to (1.5). Then, in accordance with Proposition 1.7, $(\bar{x}, \bar{y}) \in \mathrm{Sol}(MVI)$. □

Remark 2.6. Coercivity is a vital condition in Theorem 2.5. Namely, even a monotone (MVI) may have no solutions if condition (1.9) does not hold. To see this, it suffices to take $L = K = \mathbb{R}$, $F_1(x, y) = a$, $F_2(x, y) = b$, where $a \neq 0$ and $b \in \mathbb{R}$ are some constants.

We now obtain a result on the existence of saddle points of (1.1) where K and L are permitted to be unbounded.

Theorem 2.7. *Consider the minimax problem* (1.1) *and assume that* $K \subset \mathbb{R}^n$ *and* $L \subset \mathbb{R}^m$ *are nonempty closed convex subsets. If* $f(\cdot, y)$ *is pseudo-convex on* K *and* $f(x, \cdot)$ *is pseudo-concave on* L *for every* $(x, y) \in K \times L$ *and there exists a point* $(x_0, y_0) \in K \times L$ *such that* (1.9) *holds for* $F_1(u, v) = \nabla_x f(u, v)$ *and* $F_2(u, v) = \nabla_y f(u, v)$*, then* (1.1) *has a saddle point. In particular, if* $f(\cdot, y)$ *is convex on* K *and* $f(x, \cdot)$ *is concave on* L *for every fixed pair* $(x, y) \in K \times L$ *and there is a point* $(x_0, y_0) \in K \times L$ *satisfying* (1.9) *with* $F_1(u, v) = \nabla_x f(u, v)$ *and* $F_2(u, v) = \nabla_y f(u, v)$*, then* (1.1) *has a saddle point.*

Proof. By Theorem 2.5, if there exists some $(x_0, y_0) \in K \times L$ with the property (1.9) then (MVI) has a solution $(\bar{x}, \bar{y}) \in K \times L$. It remains to apply Theorem 1.2 to get the desired conclusions. □

Remark 2.8. In Theorem 2.7, the coercivity condition (1.9) is an essential assumption. Note that even a very simple minimax problem of the form (1.1) may not have solutions if there is no $(x_0, y_0) \in K \times L$ with the property (1.9). To see this, one can choose $L = K = \mathbb{R}$, $f(x, y) = ax + by$, where $a \neq 0$ and $b \in \mathbb{R}$ are some constants.

3. Pseudomonotone MVIs in Reflexive Banach Spaces

In this section we assume that X, Y are reflexive Banach spaces. The norm in the product space $X \times Y$ is given by setting $\|(x, y)\| = \|x\| + \|y\|$. Then $X \times Y$ is also a reflexive Banach space. Besides, $(X \times Y)^* \equiv X^* \times Y^*$ and the value of $(x^*, y^*) \in X^* \times Y^*$ at $(x, y) \in X \times Y$ is given by $\langle (x^*, y^*), (x, y) \rangle = \langle x^*, x \rangle + \langle y^*, y \rangle$. These conventions imply that $\|(x^*, y^*)\| = \max\{\|x^*\|, \|y^*\|\}$.

Since any bounded closed convex subset of a reflexive Banach space is weakly compact by the Banach-Alaoglu theorem, the assertions of the next theorem follow from Lemma 3.1, Theorem 3.3, and Corollary 4.7 of [19].

Theorem 3.1. (see [19]) *Suppose that* $K \subset X$ *is a nonempty closed convex subset and* $F : K \to X^*$ *is a function which is continuous on finite-dimensional subspaces, i.e., for any finite-dimensional subspace* $M \subset X$ *with* $K \cap M \neq \emptyset$ *the restricted function* $F : K \cap M \to X^*$ *is continuous from the norm topology of* $K \cap M$ *to the weak* topology of* X^**. Suppose in addition that* F *is pseudomonotone on* K*, i.e., if* $x, u \in K$ *and* $\langle F(u), x - u \rangle \geq 0$*, then* $\langle F(x), x - u \rangle \geq 0$*. Then the following holds:*

(i) *Vector* x *is a solution of the variational inequality*

$$x \in K, \quad \langle F(x), u - x \rangle \geq 0 \quad \forall u \in K \tag{3.1}$$

if and only if

$$x \in K, \quad \langle F(u), u - x \rangle \geq 0 \quad \forall u \in K.$$

(ii) *The solution set of* (3.1) *is closed and convex (may be empty).*

(iii) *If* K *is bounded, then* (3.1) *has a solution.*

(iv) *If* (3.1) *satisfies the coercivity condition, i.e., there exists* $x_0 \in K$ *such that*

$$\lim_{\substack{\|x\|\to\infty \\ x\in K}} \frac{\langle F(x)-F(x_0), x-x_0\rangle}{\|x-x_0\|} = +\infty,$$

then the problem has a solution.

Remark 3.2. As concerning assertion (iv), in [19, Corollary 4.7] it is assumed that $0 \in K$. Setting $\widetilde{K} = K - x_0$ and apply the just cited result, we get the desired claim.

An analogue of Theorem 3.1 for pseudomonotone MVIs can be formulated as follows.

Theorem 3.3. *Let* $K \subset X$, $L \subset Y$ *be nonempty closed convex subsets, and* $F_1 : K \times L \to X^*$, $F_2 : K \times L \to X^*$ *be given functions. Suppose that* F_1 *and* F_2 *are continuous on finite-dimensional subspaces and the minimax variational inequality problem* (MVI) *is pseudomonotone. Then the following holds:*

(i) *Vector* $(\bar{x}, \bar{y})$ *is a solution of* (MVI) *if and only if*

$$\langle F_1(u), u-\bar{x}\rangle - \langle F_2(v), v-\bar{y}\rangle \geq 0 \quad \forall (u,v) \in K \times L.$$

(ii) *The solution set of* (MVI) *is closed and convex (may be empty).*

(iii) *If* K *and* L *are bounded, then* (MVI) *has a solution.*

(iv) *If the problem* (MVI) *satisfies the coercivity condition, then it has a solution.*

Proof. Let $G = (F_1, -F_2)$. It is easy to show that $G : K \times L \to X^* \times Y^*$ is continuous on finite-dimensional subspaces if and only if F_1 and F_2 are continuous on finite-dimensional subspaces. Hence the assumptions made allow us to get the assertions (i)–(iv) directly from Theorem 3.1. □

The following statement on the existence of saddle points is new.

Theorem 3.4. *Consider the minimax problem* (1.1) *and assume that* $K \subset X$ *and* $L \subset Y$ *are nonempty closed convex subsets. If*

$$\begin{aligned}(3.2)\quad &\Big((x,y),(u,v) \in K \times L,\ \langle \nabla_x f(u,v), x-u\rangle - \langle \nabla_y f(u,v), y-v\rangle \geq 0\Big)\\ &\Longrightarrow \langle \nabla_x f(x,y), x-u\rangle - \langle \nabla_y f(x,y), y-v\rangle \geq 0,\end{aligned}$$

then the following holds:

(i) *Vector* $(\bar{x}, \bar{y})$ *is a saddle point of* (1.1) *if and only if*

$$\langle \nabla_x f(u,v), u-\bar{x}\rangle - \langle \nabla_y f(u,v), v-\bar{y}\rangle \geq 0 \quad \forall (u,v) \in K \times L.$$

(ii) *The set of the saddle points of* (1.1) *is closed and convex (may be empty).*

(iii) *If* K *and* L *are bounded, then* (1.1) *has a saddle point.*

(iv) *If there exists a point* $(x_0, y_0) \in K \times L$ *such that* (1.9) *holds for* $F_1(u,v) = \nabla_x f(u,v)$ *and* $F_2(u,v) = \nabla_y f(u,v)$, *then* (1.1) *has a saddle point.*

Proof. Since the pseudo-convexity of a differentiable function is equivalent to the pseudomonotonicity of its gradient mapping [3, Proposition 2.2], condition (3.2) implies that $f(\cdot, y)$ is pseudo-convex on K and $f(x, \cdot)$ is pseudo-concave on L for every $(x, y) \in K \times L$. Putting $F_1(x, y) = \nabla_x f(x, y)$, $F_2(x, y) = \nabla_y f(x, y)$, and $G(x, y) = (F_1(x, y), -F_2(x, y))$, we see that all the assumptions of Theorem 3.3 are satisfied. So the assertions (i)–(iv) of that theorem are valid. It remains to apply Theorem 1.2 to get the desired claims. □

Remark 3.5. Due to the uniqueness result recalled in Remark 1.14, if the strict pseudomonotonicity condition (1.10) is fulfilled then the MVI problem (resp., the minimax problem) considered in Theorem 3.3 (resp., in Theorem 3.4) cannot have more than one solution.

4. Strongly Monotone MVIs in Hilbert Spaces

In this section, it is assumed that X, Y are Hilbert spaces. Then X^* and Y^* can be identified with X and Y, respectively. The value of $x^* \in X^*$ at $x \in X$ is identified with the inner product $\langle x^*, x\rangle$ of two vectors in X. A similar interpretation is given for the inner product $\langle y^*, y\rangle$ of two vectors in Y. Setting $\langle (x, y), (u, v)\rangle = \langle x, u\rangle + \langle y, v\rangle$ for all $(x, y), (u, v) \in X \times Y$, we define an inner product in $X \times Y$. Note that $X \times Y$ is again a Hilbert space with the norm

$$\|(x, y)\| = (\|x\|^2 + \|y\|^2)^{1/2}.$$

We call $z \in K$ the *metric projection* of a point $x \in X$ onto a closed convex subset $K \subset X$ and write $z = P_K(x)$ if $\|x - z\| = \inf\{\|x - u\| : u \in K\}$. It is well known [10, Lemma 2.1, p. 8] that the metric projection $z = P_K(x)$ exists and is uniquely defined by x. Besides, $z = P_K(x)$ if and only if $z \in K$ and $\langle x - z, u - z\rangle \leq 0$ for every $u \in K$; see [10, Theorem 2.3, p. 9]. We also know [10, Corollary 2.4, p. 10] that $P_K(\cdot) : X \to K$ is a nonexpansive mapping, that is,

$$\|P_K(x') - P_K(x)\| \leq \|x' - x\| \quad \forall x, x' \in X.$$

It is well known that a strongly monotone VI in a Hilbert space has a unique solution. In fact, the following result implies the fundamental Lax-Milgram theorem saying that if $A : X \to X$ is a bounded linear operator and if there exists a constant $\alpha > 0$ satisfying $\langle Ax, x\rangle \geq \alpha\|x\|^2$ for all $x \in X$, then for every $u \in X$ the linear equation $Ax = u$ possesses a unique solution $x = x(u) \in X$. Since the proof of [10] only dealt with the case F is a linear operator arising from the representation of a coercive continuous bilinear form, we will use the scheme given in [5] to give a short proof for the completeness of our discussion.

Theorem 4.1. (cf. [10, Theorem 2.1, p. 24]) *Suppose that $K \subset X$ is a nonempty closed convex subset and $F : K \to X$ is a Lipschitz, strongly monotone operator, i.e., there exist constants $\ell > 0$ and $\alpha > 0$ such that*

$$\begin{aligned} &\|F(x) - F(u)\| \leq \ell\|x - u\| \quad \forall x, u \in K, \\ &\langle F(x) - F(u), x - u\rangle \geq \alpha\|x - u\|^2 \quad \forall x, u \in K. \end{aligned} \tag{4.1}$$

Then the variational inequality problem

$$\text{(4.2)}\qquad Find\ \bar{x} \in K\ \ s.t.\ \langle F(\bar{x}), x - \bar{x}\rangle \geq 0 \quad \forall x \in K$$

has a unique solution.

Proof. If K has at least two elements, then by the estimate

$$|\langle F(x) - F(u), x - u\rangle| \leq \|F(x) - F(u)\|\|x - u\|$$

and (4.1) we can assert that $\ell \geq \alpha$. Thus, there is no loss of generality in assuming that $\ell \geq \alpha$. Take any $\rho \in (0, \frac{\alpha}{\ell^2}]$ and define a map $g\colon K \to K$ by setting $g(x) = P_K(x - \rho F(x))$. Note that

$$\begin{aligned} \|g(x) - g(u)\|^2 &= \|P_K(x - \rho F(x)) - P_K(u - \rho F(u))\|^2 \\ &\leq \|(x - \rho F(x)) - (u - \rho F(u))\|^2. \end{aligned}$$

Invoking (4.1) we have

$$\begin{aligned} \|(x - \rho F(x)) - (u - \rho F(u))\|^2 &= \|x - u\|^2 - 2\rho\langle F(x) - F(u), x - u\rangle \\ &\qquad + \rho^2\|F(x) - F(u)\|^2 \\ &\leq \|x - u\|^2 - 2\rho\alpha\|x - u\|^2 + \rho^2\ell^2\|x - u\|^2 \\ &= (1 + \rho^2\ell^2 - 2\rho\alpha)\|x - u\|^2. \end{aligned}$$

Hence

$$\text{(4.3)}\qquad \|g(x) - g(u)\|^2 \leq (1 + \rho^2\ell^2 - 2\rho\alpha)\|x - u\|^2.$$

Since $0 < \rho^2 \leq \frac{\rho\alpha}{\ell^2}$, we have $\rho^2\ell^2 \leq \rho\alpha$. Hence

$$\text{(4.4)}\qquad 1 + \rho^2\ell^2 - 2\rho\alpha = (1 - \rho\alpha) + (\rho^2\ell^2 - \rho\alpha) \leq 1 - \rho\alpha.$$

As $\rho\alpha \leq \frac{\alpha^2}{\ell^2} \leq 1$, we see that $0 \leq 1 - \rho\alpha < 1$. From (4.3) and (4.4) it follows that

$$\|g(x) - g(u)\| \leq \sqrt{1 - \rho\alpha}\|x - u\| \leq \beta\|x - u\|,$$

where $\beta := \sqrt{1 - \rho\alpha} \in [0, 1)$. By the Banach contractive mapping principle, there is a unique point $\bar{x} \in K$ satisfying $g(\bar{x}) = \bar{x}$. The latter means that $P_K(\bar{x} - \rho F(\bar{x})) = \bar{x}$. Using the characterization of the metric projection recalled above, we can rewrite the last equality equivalently as follows:

$$\langle F(\bar{x}), u - \bar{x}\rangle \geq 0 \quad \forall u \in K.$$

This shows that $\bar{x}$ is a solution of (4.2). Observing that the latter holds if and only if $g(\bar{x}) = \bar{x}$, from the uniqueness of the fixed point of g we obtain the solution uniqueness of (4.2). □

Let us formulate an analogue of Theorem 4.1 for strongly monotone MVIs.

Theorem 4.2. *Suppose that $K \subset X$, $L \subset Y$ are nonempty closed convex subsets, and $F_1 : K \times L \to X$ and $F_2 : K \times L \to Y$ are such that there exist constants $\ell_i > 0$ $(i = 1, 2)$ such that*

$$\text{(4.5)}\quad \|F_i(x, y) - F_i(u, v)\| \leq \ell_i\|(x, y) - (u, v)\| \quad \forall (x, y), (u, v) \in K \times L,\ i = 1, 2.$$

If the minimax variational inequality (MVI) *is strongly monotone, then it has a unique solution* $(\bar{x}, \bar{y}) \in K \times L$.

Proof. Consider the map $G = (F_1, -F_2) : K \times L \to X^* \times Y^*$ and note that (4.5) yields

$$\|G(x,y) - G(u,v)\| \le \ell \|(x,y) - (u,v)\| \quad \forall (x,y),(u,v) \in K \times L, \tag{4.6}$$

where $\ell := \sqrt{\ell_1^2 + \ell_2^2}$. Since (MVI) is strongly monotone, there exists $\alpha > 0$ such that (1.11) holds. This means that condition (1.8) is valid for G. On the basis of (4.6) and (1.8), by Theorem 4.1 we can infer that (1.5) has a unique solution $(\bar{x}, \bar{y}) \in K \times L$. The desired conclusion now follows from applying Proposition 1.7. □

Recall that a function $\varphi\colon X \to \mathbb{R}$ is said to be *strongly convex* on a convex set $K \subset X$ if there exists $\rho > 0$ such that

$$\varphi((1-t)x + tu) \le (1-t)\varphi(x) + t\varphi(u) - \rho t(1-t)\|x-u\|^2, \quad \forall x, u \in K,\ \forall t \in (0,1).$$

The number ρ is called a *coefficient of strong convexity* of φ on K. If $-\varphi$ is strongly convex on K with a coefficient of strong convexity $\rho > 0$, then φ is said to be *strongly concave* on K with the coefficient of strong concavity $\rho > 0$. It is well known [17, Lemma 1, p. 184] that φ is strongly convex on K with a coefficient of strong convexity ρ if and only if the function $\widetilde{\varphi}(x) := \varphi(x) - \rho\|x\|^2$ is convex on K. Moreover, if φ is continuously differentiable in an open set containing K, then this strong convexity property holds if and only if

$$\langle \nabla\varphi(x) - \nabla\varphi(u), x - u \rangle \ge 2\rho\|x-u\|^2 \quad \forall x, u \in K.$$

A proof of the fact can be found in [17] for the case $X = \mathbb{R}^n$. Observe that the method of proof works also for the case where X is an arbitrary Hilbert space. (See also [18, Propositions 4.3 and 4.10], where it is also assumed that $X = \mathbb{R}^n$.)

Theorem 4.2 gives us the following result on the existence and uniqueness of a saddle point.

Theorem 4.3. *Consider the minimax problem* (1.1) *and assume that* $K \subset X$ *and* $L \subset Y$ *are nonempty closed convex subsets. If there exist constants* $\alpha > 0$ *and* $\ell_i > 0$ $(i = 1, 2)$ *such that the conditions* (1.11) *and* (4.5) *are satisfied for* $F_1(u,v) := \nabla_x f(u,v)$ *and* $F_2(u,v) := \nabla_y f(u,v)$, *then* (1.1) *has a unique saddle point* $(\bar{x}, \bar{y}) \in K \times L$.

Proof. It suffices to apply Theorems 4.2 and 1.2, observing that the assumptions made imply that, for any $(x,y) \in K \times L$, $f(\cdot, y)$ (resp., $f(x, \cdot)$) is strongly convex on K (resp., strongly concave on L) with the coefficient of strong convexity $\alpha/2$ (resp., with the coefficient of strong concavity $\alpha/2$) . □

Combining Theorem 4.3 with the analysis of Example 1.16 and noting that the Lipschitz condition (4.5) holds for the partial gradient mappings $F_1(u,v) := \nabla_x f(u,v)$ and $F_2(u,v) := \nabla_y f(u,v)$ with

$$\ell_1 := (\|A\|^2 + \|B\|^2)^{1/2} \quad \text{and} \quad \ell_2 := (\|B\|^2 + \|C\|^2)^{1/2},$$

we obtain the following result on the solution existence of quadratic minimax problems.

Theorem 4.4. *Consider the minimax problem* (1.1) *of the form described in Example 1.16. If* $u^T Au > 0$ *for all* $u \in (\operatorname{span} K) \setminus \{0\}$ *and* $v^T Cv > 0$ *for all* $v \in (\operatorname{span} L) \setminus \{0\}$, *then* (1.1) *has a unique saddle point* $(\bar{x}, \bar{y}) \in K \times L$.

5. Illustrative Examples

First, let us consider some MVIs which are resulted from twice continuously differentiable minimax problems via differentiation.

Example 5.1. Let $X = Y = \mathbb{R}$. Consider the following minimax problems and the corresponding MVIs with $F_1(u, v) = \nabla_x f(u, v)$ and $F_2(u, v) = \nabla_y f(u, v)$:

1. $K = [1, 2]$, $L = [1, 3]$, $f(x, y) = xy$.
2. $K = [1, +\infty)$, $L = [0, 3]$, $f(x, y) = x^2y^3$.
3. $K = [-1, 1]$, $L = [-1, 2]$, $f(x, y) = x^2y^2$.
4. $K = [-1, 1]$, $L = [-1, 2]$, $f(x, y) = xy$.
5. $K = (-\infty, +\infty)$, $L = [-1, 2]$, $f(x, y) = x^2y^3$.

Denote by S_i the set of the saddle points of the i-th minimax problem and by $\operatorname{Sol}(MVI)_i$ the solution set of the corresponding MVI problem. We have

$$
\begin{aligned}
&S_1 = \{(1, 3)\}, \quad \operatorname{Sol}(MVI)_1 = \{(1, 3)\},\\
&S_2 = \{(1, 3)\}, \quad \operatorname{Sol}(MVI)_2 = \big([1, +\infty) \times \{0\}\big) \cup \{(1, 3)\},\\
&S_3 = \{0\} \times [-1, 2], \quad \operatorname{Sol}(MVI)_3 = \big(\{0\} \times [-1, 2]\big) \cup \big([-1, 1] \times \{0\}\big),\\
&S_4 = \{(0, 0)\}, \quad \operatorname{Sol}(MVI)_4 = \{(0, 0)\},\\
&S_5 = \{0\} \times [0, 2], \quad \operatorname{Sol}(MVI)_5 = \big(\{0\} \times [-1, 2]\big) \cup \big((-\infty, +\infty) \times \{0\}\big).
\end{aligned}
$$

Second, we look back to the coercivity and strong monotonicity assumptions used in the preceding sections. The next example shows that a "uniform partial strong monotonicity" of F_1 and $-F_2$ together with the Lipschitz condition (4.5) is not enough for obtaining the solution existence stated in Theorem 4.2. The same example tells us that a "uniform partial coercivity" of F_1 and F_2 together with the continuity of these functions is not enough for having the conclusion of Theorem 2.5.

Example 5.2. Consider (MVI) with $K = \mathbb{R}$, $L = [1, +\infty) \subset \mathbb{R}$, and

$$F_1(x, y) = x - 2y, \quad F_2(x, y) = -y + x, \quad \forall (x, y) \in K \times L.$$

Then, $\operatorname{Sol}(MVI) = \emptyset$. Indeed, if there existed $(\bar{x}, \bar{y}) \in K \times L$ with

$$(-\bar{y} + \bar{x})(y - \bar{y}) \le 0 \le (\bar{x} - 2\bar{y})(x - \bar{x}) \quad \forall x \in \mathbb{R},\ \forall y \ge 1,$$

then we would have $\bar{x} = 2\bar{y}$ and $\bar{y}(y - \bar{y}) \le 0$ for all $y \in L$, which is impossible. It is easy to see that

$$
\begin{aligned}
&\langle F_1(x, y) - F_1(u, y), x - u\rangle = (x - u)^2 \quad \forall x, u \in K,\ \forall y \in L,\\
&-\langle F_2(x, y) - F_2(x, v), y - v\rangle = (y - v)^2 \quad \forall y, v \in L,\ \forall x \in K.
\end{aligned}
$$

Thus, a "uniform partial strong monotonicity" holds. Since

$$\begin{aligned}&\langle F_1(x,y)-F_1(u,v),x-u\rangle-\langle F_2(x,y)-F_2(u,v),y-v\rangle\\&=(x-u)^2+(y-v)^2-3(x-u)(y-v),\end{aligned}$$

one cannot find any $\alpha>0$ satisfying the strong monotonicity condition (1.11). Concerning "uniform partial coercivity", note that for any $(x_0,y_0)\in K\times L$ we have

$$\lim_{\substack{\|x\|\to\infty\\ x\in K}}\frac{\langle F_1(x,y)-F_1(x_0,y),x-x_0\rangle}{\|x-x_0\|}=\lim_{\substack{|x|\to\infty\\ x\in K}}\frac{(x-x_0)(x-x_0)}{|x-x_0|}=+\infty$$

uniformly on $y\in L$, and

$$\lim_{\substack{\|y\|\to\infty\\ y\in L}}\frac{\langle F_2(x,y)-F_2(x,y_0),y-y_0\rangle}{\|y-y_0\|}=\lim_{\substack{|y|\to\infty\\ y\in L}}\frac{-(y-y_0)(y-y_0)}{|y-y_0|}=-\infty$$

uniformly on $x\in K$. Meanwhile, the coercivity condition (1.9) fails to hold. Indeed, for any fixed point $(x_0,y_0)\in K\times L$ we have

$$\begin{aligned}\Delta(x,y,x_0,y_0)&:=\frac{\langle F_1(x,y)-F_1(x_0,y_0),x-x_0\rangle-\langle F_2(x,y)-F_2(x_0,y_0),y-y_0\rangle}{\|x-x_0\|+\|y-y_0\|}\\&=\frac{(x-x_0)^2+(y-y_0)^2-3(x-x_0)(y-y_0)}{|x-x_0|+|y-y_0|}.\end{aligned}$$

Choosing $x_k=x_0+k$, $y_k=y_0+k$ for every $k\in\mathbb{N}$, we have $\Delta(x_k,y_k,x_0,y_0)=-\frac{1}{2}k\to-\infty$ as $k\to\infty$. Thus the condition $\lim\limits_{\substack{\|(x,y)\|\to\infty\\ (x,y)\in K\times L}}\Delta(x,y,x_0,y_0)=+\infty$ is not fulfilled.

We know that a strictly monotone VI can have at most one solution. Let us show that a "partially strictly monotone" MVI may have two solutions, or more.

Example 5.3. Consider (MVI) with $L=K=[0,1]\subset\mathbb{R}$ and $F_2(x,y)=F_1(x,y)=x^2-y^2$. We have

$$\begin{aligned}\langle F_1(x,y)-F_1(u,y),x-u\rangle&=(x^2-u^2)(x-u)\\&=(x+u)(x-u)^2>0\end{aligned}$$

for all $y\in L$ and $x,u\in K$ with $x\neq u$. Besides,

$$\begin{aligned}\langle F_2(x,y)-F_2(x,v),y-v\rangle&=-(y^2-v^2)(y-v)\\&=-(y+v)(y-v)^2<0\end{aligned}$$

for all $x\in K$ and $y,v\in L$ with $y\neq v$. Hence (MVI) satisfies a "partial strict monotonicity" condition. It is not difficult to show that $\mathrm{Sol}(MVI)=\{(t,t)\,:\,t\in[0,1]\}$.

The forthcoming example shows that even a "partially strongly monotone" MVI satisfying the Lipschitz condition (4.5) may have two solutions, or more.

Example 5.4. Consider (MVI) with $L = K = [1, \gamma] \subset \mathbb{R}$, $\gamma > 1$, and $F_1(x, y) = F_2(x, y) = x^2 - y^2$. We have $\mathrm{Sol}(MVI) = \{(t, t) : t \in [1, \gamma]\}$. The calculations given in the preceding example show that

$$\langle F_1(x, y) - F_1(u, y), x - u\rangle \geq 2(x - u)^2$$

for all $y \in L$ and $x, u \in K$. In addition,

$$\langle F_2(x, y) - F_2(x, v), y - v\rangle \leq -2(y - v)^2$$

for all $x \in K$ and $y, v \in L$. Note, however, that the strong monotonicity condition (1.11) is not satisfied, because

$$\begin{aligned}&\langle F_1(x, y) - F_1(u, v), x - u\rangle - \langle F_2(x, y) - F_2(u, v), y - v\rangle\\&= [(x^2 - y^2) - (u^2 - v^2)](x - u - y + v)\\&= 0\end{aligned}$$

for $(x, y) = (\gamma, \gamma), (u, v) = (1, 1) \in K \times L$.

The pair (F_1, F_2) in each of the last three examples does not satisfy the condition

$$\frac{\partial F_1(x, y)}{\partial y} = \frac{\partial F_2(x, y)}{\partial x} \quad \forall (x, y) \in K \times L.$$

Hence, due to Clairaut's theorem, one cannot find any twice continuously differentiable function $f : \Omega \to \mathbb{R}$, where Ω is an open set containing $K \times L$, such that $F_1(u, v) = \nabla_x f(u, v)$ and $F_2(u, v) = \nabla_y f(u, v)$. This means that the corresponding MVIs cannot be obtained from twice continuously differentiable minimax problems of the form (1.1) by differentiation.

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