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ON LOCALLY NILPOTENT MAXIMAL SUBGROUPS OF THE MULTIPLICATIVE GROUP OF A DIVISION RING

BUI XUAN HAI

Abstract. Let D be a division ring with the center F and D^* be the multiplicative group of D. In this paper we study locally nilpotent maximal subgroups of D^*. We give some conditions that influence the existence of locally nilpotent maximal subgroups in division ring with infinite center. Also, it is shown that if M is a locally nilpotent maximal subgroup that is algebraic over F, then either it is the multiplicative group of some maximal subfield of D or it is center-by-locally finite. If, in addition we assume that F is finite and M is nilpotent, then the second case cannot occur, i.e. M is the multiplicative group of some maximal subfield of D.

1. Introduction

In this paper we consider a question on the existence of maximal subgroups of the multiplicative group D^* of a division ring D with the center F. In Section 2, we restrict this question to the case of a division ring algebraic over its infinite center, otherwise, D would be commutative by Jacobson Theorem [4, p. 219]. A class of examples is given where D^* does not admit neither maximal subgroups nor locally nilpotent maximal subgroups. Also, here we investigate some properties which influence the existence of maximal subgroups.

Throughout this paper, we use the standard symbols and notation. In particular, if $S \subseteq D$ is a nonempty subset of a division ring D then $C_D(S)$ denotes the centralizer of S in D, i.e.

$$C_D(S) = \{x \in D |\ xa = ax \text{ for all } a \in S\}.$$

If $F \subseteq K$ is a field extension and $a \in K$ is an algebraic element over F, then we denote by $\min(F, a)$ the minimal polynomial of a, i.e. the irreducible monic polynomial in $F[X]$ which has a as its root. If every element of D is algebraic over its center F, then we say that D is *algebraic* over F. An element a in D is said to be *radical* over F if there exists some positive integer $n(a)$ depending on a such that $a^{n(a)} \in F$. A subset $S \subseteq D$ is *radical* over F if every its element is radical over F. If G is a group, then the center of G is denoted by $Z(G)$. Finally,

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in this paper "a normal maximal subgroup" means "a maximal subgroup which is normal". Similarly for other constructions: for example, "a locally nilpotent maximal subgroup" means "a maximal subgroup which is locally nilpotent", etc...

2. A division ring with infinite center

In the first we consider a special class of division rings including the division ring of real quaternions. In fact, for a division ring D belonging to this class we assume that D contains some algebraic closure of its center.

Lemma 2.1. *Let D be a division ring which is algebraic over its center F and suppose that D contains an algebraic closure L of F. Then, for any element a in D, there exists some element b in D^* such that $bab^{-1} \in L$.*

Proof. Suppose that $a \in D$ is arbitrary. Denote by $min(F, a)$ the minimal polynomial of a over F. Since L is an algebraic closure of $F, min(F, a)$ has some root, say ω in L. Thus, a and ω have the same minimal polynomial over F. By Dickson's Theorem [4, p. 265], there exists some element b in D^* such that $bab^{-1} = \omega \in L$. □

Theorem 2.1. *If D is a division ring as in Lemma 2.1, then D^* has no normal maximal subgroups.*

Proof. Let $L \subseteq D$ be an algebraic closure of F and suppose that M is a normal maximal subgroup of D^*. Then, $D^*/M \simeq \mathbb{Z}_p$ for some prime number p. Consider an arbitrary element a in D^*. By Lemma 2.1, there exists some element b in D^* such that $bab^{-1} \in L$. Since L is an algebraic closure of F, the polynomial $f(X) = X^p - bab^{-1} \in L[X]$ has some root c in L; hence

$$f(c) = c^p - bab^{-1} = 0.$$

It follows that $a = b^{-1}c^pb = (b^{-1}cb)^p \in M$. Hence, $D^* = M$ that is a contradiction. □

Since the division ring H of real quaternions satisfies the supposition of Lemma 2.1 above, the result obtained in Theorem 2.1 generalizes Theorem 13 in [1].

As some interesting application of this theorem, we shall obtain again a series of fields (including the field of complex numbers) whose multiplicative groups contain no maximal subgroups.

Corollary 2.1. *Every algebraically closed field contains no maximal subgroups.*

This conclusion does not need Theorem 2.1. One can see it directly as follows. The multiplicative group of an algebraically closed field is obviously a divisible abelian group by the solvability of polynomials $X^n - a$. Since divisible abelian groups cannot contain maximal subgroups, the corollary follows.

Notice that from the proof of Theorem 2.1, it follows easily that multiplicative groups of the considered division rings are (not necessarily abelian) divisible groups. This observation is calling author's attention to the following question: are these groups always complete?

Lemma 2.2. *Suppose that a finite field extension $F \subset K$ does not have proper intermediate subfields. Then:*

(i) *either K is separable over F, or*

(ii) *K is purely inseparable over F. Moreover, in this case,* $\mathrm{char}F = p > 0, K$ *is radical over F and $[K : F] = p$.*

Proof. If there exists a separable over F element a in $K \setminus F$, then $K = F(a)$ is separable over F. Suppose that every element of $K \setminus F$ is inseparable over F. Then K is purely inseparable over F. Clearly, in this case, we have $charF = p > 0$. Now, consider some element a in $K \setminus F$. We can find some positive integer $n = n(a)$ depending on a such that $a^{p^n} \in F$. Suppose that n is a minimal positive integer such that $a^{p^n} \in F$. Setting $b = a^{p^{n-1}}$, we have $b^p \in F$ and $b \notin F$. It follows that $K = F(b)$ and $min(F, b) = X^p - b^p$, so $[K : F] = p$. □

Theorem 2.2. *If D is a division ring as in Lemma 2.1, then D^* contains no locally nilpotent maximal subgroups.*

Proof. If D is a field, then by Corollary 2.1, D contains no maximal subgroups. Now, suppose that D is non-commutative and M is a locally nilpotent maximal subgroup of D^*. By [3, Th. 3.2], M is the multiplicative group of some maximal subfield K of D. Moreover, M contains F. Since D is non-commutative, $K \neq F$ and by [2, Th. 1], there are no proper intermediate subfields of the field extension $F \subset K$ and $Gal(K/F) = \{Id_K\}$. If $a \in K \setminus F$, then $K = F(a)$. Since a is algebraic over F, it follows that $[K : F] = [F(a) : F] < \infty$. By Lemma 2.1, there exists some element b in D^* such that $bab^{-1} \in L$ (L is an algebraic closure of F, lying in D). Therefore, $bKb^{-1} = bF(a)b^{-1} \subseteq L$ and $bK^*b^{-1} \subseteq L^*$. Since $K^* = M$ is maximal in D^*, bK^*b^{-1} is maximal in D^*. This forces $L^* = bK^*b^{-1}$ and consequently $L = bKb^{-1}$. Hence, one can suppose that $K = L$. In particular, it follows that $F \subset K$ is a normal extension.

By Lemma 2.2, either K is separable over F or K is radical over F and $[K : F] = p = charF > 0$. In the first case, since K is normal over F, it follows that $F \subset K$ is a Galois extension. Therefore, $|Gal(K/F)| = [K : F] \neq 1$, that is a contradiction. In the last case, for any $u \in D^*$, there exists $v \in D^*$ such that $vuv^{-1} \in L = K$; hence $u \in v^{-1}Kv$. Since K is radical over F, u is radical over F too. Thus, we have proved that D is radical over F. Now, by Kaplansky's Theorem (see [4, p. 259]), D is commutative, that is again a contradiction. □

Note that in [2], it was proved that the division ring of real quaternions does not contain nilpotent maximal subgroups. So, the theorem we have proved strongly generalizes this result.

Now, suppose that M is a maximal subgroup of D^* and P is the prime subfield of F. Denote by $P(Z(M))$ the subfield of D generated by $Z(M)$, i.e. $P(Z(M))$ is the minimal subfield of D containing $Z(M)$. It was proved in [2] that $F \subseteq$

$P(Z(M))$. Moreover, if F is infinite then $Z(M) = M \cap F$ if and only if $P(Z(M)) = F$ (see [2, Prop. 1]). Using this fact we can prove the following result.

Proposition 2.1. *Let D be a non-commutative division ring with infinite center F. Then the following conditions are equivalent:*

(i) *$Z(M) = M \cap F$ for every maximal subgroup M of D^*.*

(ii) *D^* contains no maximal subgroups that are multiplicative groups of some division subrings of D.*

Proof. Suppose that (i) holds and M is a maximal subgroup of D such that $K := M \cup \{0\}$ is a division subring of D. By [2, Prop. 1], $F = P(Z(M)) = P(Z(K)) = Z(K)$. By [2, Lem. 6], K^* is self-normalized in D^*. So $C_D(K) = Z(K)$, hence $C_D(K) = F$. By Double Centralizer Theorem we have $C_D(C_D(K)) = K$. It follows that $K = C_D(C_D(K)) = C_D(F) = D$, that is a contradiction in view of the maximality of $M = K^*$ in D^*.

Conversely, suppose that D^* contains no maximal subgroups that are multiplicative groups of some division subrings of D and M is a maximal subgroup of D. By setting $K := M \cup \{0\}$ we have $K \subseteq C_D(Z(M))$. So by maximality of $K^* := M$ in D^*, either $C_D(Z(M))^* = K^*$ or $C_D(Z(M))^* = D^*$. Since by supposition, K is not a division subring, we have $C_D(Z(M))^* = D^*$; hence, $Z(M) \subseteq F$ and consequently $Z(M) = M \cap F$. □

Corollary 2.2. *Let D be a non-commutative division ring that is algebraic over its center F. If $Z(M) = M \cap F$ for every maximal subgroup M of D^*, then D contains no locally nilpotent maximal subgroups.*

Proof. Suppose that $Z(M) = M \cap F$ for every maximal subgroup M of D^*. If M is a locally nilpotent maximal subgroup of D^*, then by [3, Th. 3.2], $M \cup \{0\}$ is the maximal subfield of D that is a contradiction to the conclusion of Proposition 2.1 above. □

Note that in [2], it was proved that in the division ring of real quaternions for every maximal subgroup M, we have $Z(M) = M \cap F$. Hence, in view of Proposition 2.1 and Corollary 2.2 above we have the following corollary.

Corollary 2.3. *The division ring H of real quaternions contains no maximal subgroups that are multiplicative groups of some division subrings of H. Also, H contains no locally nilpotent maximal subgroups.*

Note that the last assertion of this corollary could also follow from Theorem 2.2 above.

3. A division ring with finite center

Let D be a non-commutative division ring with finite center F. In this section, we give some characterization of nilpotent maximal subgroups of D^* that are algebraic over F. Recall that a group G is said to be *center-by-locally finite* if $G/Z(G)$ is a locally finite group.

Lemma 3.1. *Let D be a non-commutative division ring with center F and suppose that M is a locally nilpotent maximal subgroup of D^* that is algebraic over F. Then:*

(i) *either $F^* \subseteq M$ and there exists a maximal subfield K of D such that $M = K^*$ or,*

(ii) *M is center-by-locally finite.*

Proof. Since M is maximal in D^*, either $M = F(M)^*$ or $F(M) = D$. If $M = F(M)^*$, then $F^* \subseteq M$ and by [3, Th. 2.2], $K = F(M)$ is the maximal subfield of D.

Now, suppose that $F(M) = D$. Since M is algebraic over F, we have $D = F(M) = F[M]$; so M is absolutely irreducible. By [5, Th. 5.7.11, p. 215], M is center-by-locally finite. □

Now, we are ready to prove the following result for a division ring with finite center.

Theorem 3.1. *Let D be a non-commutative division ring with center F and suppose that M is a nilpotent maximal subgroup of D^* that is algebraic over F. If F is finite, then M is the multiplicative group of some maximal subfield of D.*

Proof. Suppose that F is finite. In view of Lemma 3.1, it suffices to show that the case (ii) cannot occur. Thus, suppose that $F(M) = D$. Since M is maximal in D^*, we can show that $F^* \subseteq M$. If not, we have $F^*M = D^*$, so $D' = M' \subseteq M$. Then D^* is solvable and by Hua's Theorem (see, for example [4, p. 223]) it follows that D is commutative, that is a contradiction. Therefore $F^* \subseteq M$, so $F^* = Z(M)$. By Lemma 3.1 (ii), M/F^* is locally finite. Consider arbitrary elements x, y in M. Then, the subgroup $\langle xF^*, yF^* \rangle$ of M/F^* generated by xF^* and yF^* is finite. Suppose that g is the restriction of the natural homomorphism $M \longrightarrow M/F^*$ on the subgroup $\langle x, y \rangle$. Then, we have $Ker g = \langle x, y \rangle \cap F^*$ and $Im g = \langle xF^*, yF^* \rangle$. Since F^* and $\langle xF^*, yF^* \rangle$ are both finite, it follows that $\langle x, y \rangle$ is finite. Therefore, $\langle x, y \rangle$ is cyclic and in particular, x, y commute with each other. So, M is abelian and consequently, $D = F(M)$ is commutative, that is a contradiction. □

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References

[1] S. Akbari and M. Mahdavi-Hezavehi, On the existence of normal maximal subgroups in division rings, *J. Pure Appl. Algebra* **171** (2002), 123–131.

[2] B. X. Hai and L. K. Huynh, On maximal subgroups in division rings, *Acta Math. Vietnam.* **30** (1) (2005), 35–43.

[3] B. X. Hai and N. V. Thin, On locally nilpotent subgroups of $GL_1(D)$, *Communications in Algebra* **37** (2009), 712–718.

[4] T. Y. Lam, *A First Course In Non-Commutative Rings*, GMT 131, Springer-Verlag, 1991.

[5] M. Shirvani and B. A. F. Wehrfritz, *Skew Linear Groups*, London Mathematical Society, Lecture Note Series 118, Cambridge University Press, 1986.

Faculty of Mathematics and Computer Science
University of Science, VNU - HCM City
227 Nguyen Van Cu str., Dist. 5, Ho Chi Minh City, Vietnam

E-mail address: bxhai@hcmus.edu.vn