ACTA MATHEMATICA VIETNAMICA
Volume 36, Number 1, 2011, pp. 73–87

ON THE RATIONAL RECURSIVE SEQUENCE $x_{n+1} = \dfrac{A + \alpha_0 x_n + \alpha_1 x_{n-\sigma}}{B + \beta_0 x_n + \beta_1 x_{n-\tau}}$

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Abstract. The main objective of this paper is to study the boundedness, the periodicity, the convergence and the global stability of the positive solutions of the difference equation

$$x_{n+1} = \frac{A + \alpha_0 x_n + \alpha_1 x_{n-\sigma}}{B + \beta_0 x_n + \beta_1 x_{n-\tau}}, \qquad n = 0, 1, 2, ...$$

where the coefficients $A, B, \alpha_0, \alpha_1, \beta_0, \beta_1 \in (0, \infty)$ and $\sigma, \tau \in N$. The initial conditions $x_{-\omega}$,...,x_{-1}, x_0 are arbitrary positive real numbers and $\omega = \max\{\tau, \sigma\}$. Some numerical examples are presented.

1. Introduction

Our goal in this paper is to investigate the boundedness, the periodicity, the convergence and the global stability of the positive solutions of the difference equation

$$x_{n+1} = \frac{A + \alpha_0 x_n + \alpha_1 x_{n-\sigma}}{B + \beta_0 x_n + \beta_1 x_{n-\tau}}, \qquad n = 0, 1, 2, ... \tag{1.1}$$

where the coefficients $A, B, \alpha_0, \alpha_1, \beta_0, \beta_1 \in (0, \infty)$ and $\sigma, \tau \in N$. The initial conditions $x_{-\omega}, ..., x_{-1}, x_0$ are arbitrary positive real numbers and $\omega = \max\{\tau, \sigma\}$. The case where any of $A, B, \alpha_0, \alpha_1, \beta_0, \beta_1$ is allowed to be zero gives different special cases of the equation (1.1) which are studied by many authors, (see for example ([1]-[16]). For the related work, see [17]-[40]. The study of these equations is challenging and rewarding and is still in its infancy. We believe that the nonlinear rational difference equations are of importance in their own right. Furthermore, the results about such equations offer prototypes for the study of the global behavior of nonlinear difference equations. Note that the difference equation (1.1) has been discussed in [24] when $\sigma = \tau = 1$.

Definition 1. A difference equation of order $(\omega + 1)$ is of the form

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-\omega}), \qquad n = 0, 1, 2, ... \tag{1.2}$$

Received September 21, 2009; in revised form June 2, 2010.
2000 *Mathematics Subject Classification.* 39A10, 39A11, 39A99, 34C99.
Key words and phrases. Difference equations, linearized, boundedness, global stability, period two solutions.

where F is a continuous function which maps some set $J^{\omega+1}$ into J and J is a set of real numbers. An equilibrium point $\widetilde{x}$ of this equation is a point that satisfies the condition $\widetilde{x} = F(\widetilde{x}, \widetilde{x}, ..., \widetilde{x})$. That is, the constant sequence $\{x_n\}_{n=-\omega}^{\infty}$ with $x_n = \widetilde{x}$ $for \quad all \quad n \geq -\omega$ is a solution of that equation.

Definition 2. Let $\widetilde{x} \in (0, \infty)$ be an equilibrium point of the difference equation (1.2). Then

(i) An equilibrium point $\widetilde{x}$ of the difference equation (1.2) is called locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $x_{-\omega}, ..., x_{-1}, x_0 \in (0, \infty)$ with $|x_{-\omega} - \widetilde{x}| + ... + |x_{-1} - \widetilde{x}| + |x_0 - \widetilde{x}| < \delta$, then $|x_n - \widetilde{x}| < \varepsilon$ for all $n \geq -\omega$.

(ii) An equilibrium point $\widetilde{x}$ of the difference equation (1.2) is called locally asymptotically stable if it is locally stable and there exists $\gamma > 0$ such that, if $x_{-\omega}, ..., x_{-1}, x_0 \in (0, \infty)$ with $|x_{-\omega} - \widetilde{x}| + ... + |x_{-1} - \widetilde{x}| + |x_0 - \widetilde{x}| < \gamma$, then

$$\lim_{n \to \infty} x_n = \widetilde{x}.$$

(iii) An equilibrium point $\widetilde{x}$ of the difference equation (1.2) is called a global attractor if for every $x_{-\omega}, ..., x_{-1}, x_0 \in (0, \infty)$ we have

$$\lim_{n \to \infty} x_n = \widetilde{x}.$$

(iv) An equilibrium point $\widetilde{x}$ of the equation (1.2) is called globally asymptotically stable if it is locally stable and a global attractor.

(v) An equilibrium point $\widetilde{x}$ of the difference equation (1.2) is called unstable if it is not locally stable.

Definition 3. We say that a sequence $\{x_n\}_{n=-\omega}^{\infty}$ is bounded and persisting if there exist positive constants m and M such that

$$m \leq x_n \leq M \quad \text{for all} \quad n \geq -\omega.$$

Definition 4. A sequence $\{x_n\}_{n=-\omega}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -\omega$. A sequence $\{x_n\}_{n=-\omega}^{\infty}$ is said to be periodic with prime period p if p is the smallest positive integer having this property.

Assume that $\widetilde{a} = \alpha_0 + \alpha_1$, $\overline{a} = \alpha_0 - \alpha_1$, $\widetilde{b} = \beta_0 + \beta_1$ and $\overline{b} = \beta_0 - \beta_1$. Then the equilibrium point $\widetilde{x}$ of the difference equation (1.1) is the solution of the equation

$$\widetilde{x} = (A + \widetilde{a}\,\widetilde{x}) / (B + \widetilde{b}\,\widetilde{x}). \tag{1.3}$$

Consequently, the positive equilibrium point $\widetilde{x}$ of the difference equation (1.1) is given by

$$\widetilde{x} = \left((\widetilde{a} - B) + \sqrt{(\widetilde{a} - B)^2 + 4A\widetilde{b}} \right) / 2\widetilde{b}.$$

Let $F : (0, \infty)^3 \longrightarrow (0, \infty)$ be a continuous function defined by

$$F(u_0, u_1, u_2) = \frac{A + \alpha_0 u_0 + \alpha_1 u_1}{B + \beta_0 u_0 + \beta_1 u_2}. \tag{1.4}$$

Then the linearized equation associated with the difference equation (1.1) about the positive equilibrium point $\widetilde{x}$ takes the form

$$\begin{aligned} y_{n+1} &= \frac{\partial F(\widetilde{x},\widetilde{x},\widetilde{x})}{\partial u_0} y_n + \frac{\partial F(\widetilde{x},\widetilde{x},\widetilde{x})}{\partial u_1} y_{n-\sigma} + \frac{\partial F(\widetilde{x},\widetilde{x},\widetilde{x})}{\partial u_2} y_{n-\tau} \\ &= a_2 y_n + a_1 y_{n-\sigma} + a_0 y_{n-\tau}, \end{aligned} \tag{1.5}$$

where

$$a_2 = \frac{\alpha_0 - \beta_0 \widetilde{x}}{B + \widetilde{b\widetilde{x}}}, \quad a_1 = \frac{\alpha_1}{B + \widetilde{b\widetilde{x}}}, \text{ and } \quad a_0 = \frac{-\beta_1 \widetilde{x}}{B + \widetilde{b\widetilde{x}}}. \tag{1.6}$$

The characteristic equation of the linearized equation (1.5) is

$$\lambda^{n+1} = a_2 \lambda^n + a_1 \lambda^{n-\sigma} + a_0 \lambda^{n-\tau}. \tag{1.7}$$

2. Main results

In this section, we establish some results which show that the positive equilibrium point $\widetilde{x}$ of the difference equation (1.1) is globally asymptotically stable and every positive solution of the difference equation (1.1) is bounded and has prime period two.

Theorem 1. ([17, 18] The linearized stability theorem) *Suppose F is a continuously differentiable function defined on an open neighborhood of the equilibrium $\widetilde{x}$. Then the following statements are true.*

(i) *If all roots of the characteristic equation* (1.7) *of the linearized equation* (1.5) *have absolute value less than one, then the equilibrium point $\widetilde{x}$ is locally asymptotically stable.*

(ii) *If at least one root of equation* (1.7) *has absolute value greater than one, then the equilibrium point $\widetilde{x}$ is unstable.*

(iii) *If all roots of equation* (1.7) *have absolute value greater than one, then the equilibrium point $\widetilde{x}$ is a source.*

Theorem 2. (see [4, 20]) *Assume that $a, b \in R$ and $k \in N$. Then*

$$|a| + |b| < 1 \tag{2.1}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + a x_n + b x_{n-k} = 0, \quad n = 0, 1, ... \tag{2.2}$$

Remark 1. (see [10, 20]) Theorem 2 can be easily extended to a general linear difference equation of the form

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \quad n = 0, 1, 2, ... \tag{2.3}$$

where $p_1, p_2, \ldots, p_k \in R$ and $k \in \{1, 2, \ldots\}$. Then the equation (2.2) is asymptotically stable provided that

$$\sum_{i=1}^{k} |p_i| < 1. \tag{2.4}$$

Theorem 3. *Let* $\{x_n\}_{n=-\omega}^{\infty}$ *be a solution of the difference equation* (1.1) *such that for some* $n_0 \geq 0$,

$$\text{(2.5)} \qquad either \quad x_n \ \geq \ \widetilde{x} \quad for \quad n \geq n_0 + \omega,$$

$$\text{(2.6)} \qquad or \quad x_n \ \leq \ \widetilde{x} \quad for \quad n \geq n_0 + \omega,$$

where $\omega = \max\{\tau, \sigma\}$. *Then* $\{x_n\}$ *converges to* $\widetilde{x}$ *as* $n \to \infty$.

Proof. Assume that (2.5) holds. The case where (2.6) holds is similar and will be omitted. Then for $n \geq n_0 + \omega$, where $\omega = \max\{\tau, \sigma\}$, we deduce that

$$\begin{aligned} x_{n+1} &= (A + \alpha_0 x_n + \alpha_1 x_{n-\sigma}) \,/\, (B + \beta_0 x_n + \beta_1 x_{n-\tau}) \\ &= (\alpha_0 x_n + \alpha_1 x_{n-\sigma}) \left(\left(1 + \frac{A}{\alpha_0 x_n + \alpha_1 x_{n-\sigma}} \right) / \, (B + \beta_0 x_n + \beta_1 x_{n-\tau}) \right) \\ &\leq (\alpha_0 x_n + \alpha_1 x_{n-\sigma}) \frac{[1 + (A/\widetilde{a}\ \widetilde{x})]}{\left(B + \widetilde{b}\ \widetilde{x}\right)} = (\alpha_0 x_n + \alpha_1 x_{n-\sigma}) \frac{(A + \widetilde{a}\ \widetilde{x})}{\widetilde{a}\ \widetilde{x} \left(B + \widetilde{b}\ \widetilde{x}\right)}. \end{aligned}$$

With the aid of (1.3), the last inequality becomes

$$x_{n+1} \leq \ (\alpha_0 x_n + \alpha_1 x_{n-\sigma}) \,/\, \widetilde{a},$$

and so

$$\text{(2.7)} \qquad x_{n+1} \leq \max_{0 \leq i \leq \omega} \{x_{n-i}\} \quad \text{for} \quad n \geq n_0 + \omega.$$

Set

$$\text{(2.8)} \qquad y_n = \max_{0 \leq i \leq \omega} \{x_{n-i}\} \quad \text{for} \quad n \geq n_0 + \omega.$$

Then clearly

$$\text{(2.9)} \qquad y_n \geq x_{n+1} \geq \widetilde{x} \quad \text{for} \quad n \geq n_0 + \omega.$$

Next we claim that

$$\text{(2.10)} \qquad y_{n+1} \leq \ y_n \quad \text{for} \quad n \geq n_0 + \omega.$$

Now, we have

$$y_{n+1} = \max_{0 \leq i \leq \omega} \{x_{n+1-i}\} = \max \left\{ x_{n+1}, \max_{0 \leq i \leq \omega - 1} \{x_{n-i}\} \right\} \leq \max\{x_{n+1}, \ y_n\} = \ y_n.$$

From (2.9) and (2.10), it follows that the sequence $\{y_n\}$ is convergent and that

$$\text{(2.11)} \qquad y = \lim_{n \to \infty} y_n \geq \widetilde{x}.$$

Furthermore, we get

$$x_{n+1} \leq (A + \alpha_0 x_n + \alpha_1 x_{n-\sigma}) \,/ \left(B + \widetilde{b}\ \widetilde{x}\right) \leq (A + \widetilde{a}\ y_n) \,/ \left(B + \widetilde{b}\ \widetilde{x}\right).$$

From this and by using (2.10), we obtain

$$x_{n+i} \leq (A + \widetilde{a}\ y_{n+i-1}) \,/ \left(B + \widetilde{b}\ \widetilde{x}\right) \leq (A + \widetilde{a}\ y_n) \,/ \left(B + \widetilde{b}\ \widetilde{x}\right) \text{ for } i = 1, ..., \omega + 1.$$

Then

$$y_{n+\omega+1} = \max_{1\leq i\leq \omega+1} \{x_{n+i}\} \leq (A + \widetilde{a}\; y_n) / \left(B + \widetilde{b}\; \widetilde{x}\right), \tag{2.12}$$

and by letting $n \longrightarrow \infty$, we obtain

$$y \leq \frac{A + \widetilde{a}\; y}{B + \widetilde{b}\; \widetilde{x}}. \tag{2.13}$$

Subtracting (1.3) from (2.13), we have the inequality

$$(y - \widetilde{x}) \left(1 - \frac{\widetilde{a}}{B + \widetilde{b}\; \widetilde{x}}\right) \leq 0. \tag{2.14}$$

Since $\widetilde{x} > \frac{\widetilde{a}-B}{\widetilde{b}}$, then the term $\left(1 - \frac{\widetilde{a}}{B+\widetilde{b}\; \widetilde{x}}\right)$ is positive. Consequently, we deduce that $y \leq \widetilde{x}$, and in view of (2.11) we obtain $y = \widetilde{x}$. Thus, the proof of Theorem 3 is complete. □

Theorem 4. *Let $\{x_n\}_{n=-\omega}^{\infty}$ be a positive solution of the difference equation* (1.1) *and let $B > 1$. Then there exist positive constants m and M such that*

$$m \leq x_n \leq M, \quad n = 0, 1, \ldots \tag{2.15}$$

Proof. From the difference equation (1.1) we deduce for $B > 1$ that

$$x_{n+1} \leq \frac{A}{B} + \frac{1}{B} (\alpha_0 x_n + \alpha_1 x_{n-\sigma}), \quad n = 0, 1, \ldots \tag{2.16}$$

Consider the linear difference equation

$$y_{n+1} = \frac{A}{B} + \frac{1}{B} (\alpha_0 y_n + \alpha_1 y_{n-\sigma}), \quad n = 0, 1, \ldots \tag{2.17}$$

with the initial conditions $y_i = x_i > 0,\ i = -\omega, \ldots, -1, 0$. It follows by induction that

$$x_n \leq y_n. \tag{2.18}$$

First of all, assume that $B > \widetilde{a}$. Then we have $A/(B - \widetilde{a})$ is a particular solution of the equation (2.17) and every solution of the homogeneous equation which is associated with the equation (2.17) tends to zero as $n \to \infty$. Hence

$$\lim_{n\to\infty} y_n = \frac{A}{B - \widetilde{a}}.$$

From this and (2.18), it follows that the sequence $\{x_n\}$ is bounded from above by a positive constant M, say. That is,

$$x_n \leq M, \quad n = 0, 1, ...$$

Set

$$m = \frac{A}{B + \widetilde{b}M}.$$

Then, we have

$$x_{n+1} = \frac{A + \alpha_0 x_n + \alpha_1 x_{n-\sigma}}{B + \beta_0 x_n + \beta_1 x_{n-\tau}} \geq \frac{A}{B + \widetilde{b}M} = m$$

and consequently, we get

$$m \leq x_n \leq M, \quad n = 0, 1, ...$$

which completes the proof of Theorem 4 when $B > \widetilde{a}$. Secondly, consider the case when $B \leq \widetilde{a}$, it suffices to show that, $\{x_n\}$ is bounded from above by some positive constant. Assume the contrary, that $\{x_n\}$ is unbounded, then there exists a subsequence $\{x_{n_j}\}$ such that

$$\lim_{j\to\infty} n_j = \infty \quad \text{and} \quad \lim_{j\to\infty} x_{1+n_j} = \infty,$$

and

$$x_{1+n_j} = \max\{x_n : -\omega \leq n \leq 1 + n_j\}, \quad (j = 0, 1, 2, ...).$$

From (2.16), we deduce that

$$\alpha_0 x_{n_j} + \alpha_1 x_{n_j-\sigma} \geq B x_{1+n_j} - A.$$

Taking the limit as $j \to \infty$ of both sides of the last inequality, we obtain

$$\lim_{j\to\infty} \left(\alpha_0 x_{n_j} + \alpha_1 x_{n_j-\sigma}\right) = \infty. \tag{2.19}$$

It is easy to show that $x_{n_j} \leq x_{1+n_j}$ and $x_{n_j-\sigma} \leq x_{1+n_j}$ for all $\sigma \in N$. Then, as $\widetilde{a} = \alpha_0 + \alpha_1$, we have

$$\left(\alpha_0 x_{n_j} + \alpha_1 x_{n_j-\sigma}\right) \leq \widetilde{a} x_{1+n_j}. \tag{2.20}$$

From the inequality (2.20) and the difference equation (1.1), we obtain

$$\widetilde{a}A + \left(\alpha_0 x_{n_j} + \alpha_1 x_{n_j-\sigma}\right)\left[\widetilde{a} - B - \left(\beta_0 x_{n_j} + \beta_1 x_{n_j-\tau}\right)\right] \geq 0. \tag{2.21}$$

From (2.19) and (2.21), it follows that

$$\beta_0 x_{n_j} + \beta_1 x_{n_j-\tau} \leq \widetilde{a} - B. \tag{2.22}$$

Then, from (2.22) we deduce, for every $\tau \in N$ for which β_0 and β_1 are positive constants, that the subsequences $\{x_{n_j}\}$ and $\{x_{n_j-\tau}\}$ are bounded which implies that the sequence $\{\alpha_0 x_{n_j} + \alpha_1 x_{n_j-\sigma}\}$ is bounded for all $\sigma \in N$ for which α_0 and α_1 are positive constants. This contradicts (2.19) and the proof of Theorem 4 is complete. $\square$

Theorem 5. *Assume that $B > \widetilde{a}$ holds. Then the positive equilibrium point $\widetilde{x}$ of the difference equation* (1.1) *is globally asymptotically stable.*

Proof. The linearized equation (1.5) can be written in the form

$$y_{n+1} + \left(\frac{\beta_0\widetilde{x} - \alpha_0}{B + \widetilde{b}\widetilde{x}}\right) y_n - \left(\frac{\alpha_1}{B + \widetilde{b}\widetilde{x}}\right) y_{n-\sigma} + \left(\frac{\beta_1\widetilde{x}}{B + \widetilde{b}\widetilde{x}}\right) y_{n-\tau} = 0.$$

As $B > \widetilde{a}$, we get

$$\left|\frac{\beta_0\widetilde{x} - \alpha_0}{B + \widetilde{b}\widetilde{x}}\right| + \left|\frac{-\alpha_1}{B + \widetilde{b}\widetilde{x}}\right| + \left|\frac{\beta_1\widetilde{x}}{B + \widetilde{b}\widetilde{x}}\right| \leq \frac{\widetilde{a} + \widetilde{b}\,\widetilde{x}}{B + \widetilde{b}\,\widetilde{x}} < 1.$$

Thus, by Theorems 1, 2, we deduce that the equilibrium point $\widetilde{x}$ of the difference equation (1.1) is locally asymptotically stable. It remains to prove that the equilibrium point $\widetilde{x}$ is a global attractor. To this end, set $I = \lim_{n\to\infty} \inf x_n$

and $S = \lim_{n\to\infty} \sup x_n$, which by Theorem 4 exist and are positive numbers. Then, from the difference equation (1.1), we see that

$$S \leq \frac{A + \widetilde{a}S}{B + \widetilde{b}I} \quad \text{and} \quad I \geq \frac{A + \widetilde{a}I}{B + \widetilde{b}S}.$$

Hence,

$$A + (\widetilde{a} - B)\, I \leq \widetilde{b}IS \leq A + (\widetilde{a} \; - B)\, S.$$

From which, it follows that $I \geq S$. Thus, we have $I = S$. The proof of Theorem 5 is now complete. □

Theorem 6. (i) *If either σ and τ are even positive integers or σ is a positive odd integer and τ is a positive even integer, then the difference equation* (1.1) *has no positive solutions of prime period two.*

(ii) *If either σ and τ are odd positive integers or σ is a positive even integer and τ is a positive odd integer, then the necessary and sufficient condition for the difference equation* (1.1) *to have positive solutions of prime period two is that the inequality*

$$4\beta_1 \left[A\beta_1 - \alpha_0 \left(B + \overline{a}\right)\right] < \overline{b} \left(B + \overline{a}\right)^2, \tag{2.23}$$

is valid, provided that $B + \overline{a} < 0$ and $\overline{b} > 0$.

Proof. Suppose that there exist positive distinct solutions of prime period two

$$......., P, Q, P, Q, ...$$

of the difference equation (1.1), now, we discuss the following cases:

Case 1: σ and τ are even positive integers. In this case, $x_n = x_{n-\sigma} = x_{n-\tau}$. Then there exists a positive period two solution $\{x_n\}$ such that

$$\begin{aligned} x_{2k} &= P, \quad k = -1, 0, 1, ... \\ x_{2k+1} &= Q, \quad k = -1, 0, 1, ... \end{aligned}$$

and $P \neq Q$. From the difference equation (1.1), we have

$$P = \frac{A + (\alpha_0 + \alpha_1)\, Q}{B + (\beta_0 + \beta_1)\, Q}, \qquad Q = \frac{A + (\alpha_0 + \alpha_1)\, P}{B + (\beta_0 + \beta_1)\, P}.$$

Consequently, we obtain

$$A + (\alpha_0 + \alpha_1)\, Q = BP + (\beta_0 + \beta_1)\, PQ, \tag{2.24}$$

and

$$A + (\alpha_0 + \alpha_1)\, P = BQ + (\beta_0 + \beta_1)\, PQ. \tag{2.25}$$

By subtracting, we have

$$(\alpha_0 + \alpha_1 + B)\;\; (P - Q) = 0. \tag{2.26}$$

This implies $P = Q$. This is a contradiction. Thus, equation (1.1) has no prime period two solution.

Case 2: σ is a positive odd integer and τ is a positive even integer. In this case, $x_{n+1} = x_{n-\sigma}$ and $x_n = x_{n-\tau}$. From the difference equation (1.1), we have

$$P = \frac{A + \alpha_0 Q + \alpha_1 P}{B + (\beta_0 + \beta_1) Q}, \qquad Q = \frac{A + \alpha_0 P + \alpha_1 Q}{B + (\beta_0 + \beta_1) P}.$$

Consequently, we obtain

$$A + \alpha_0 Q + \alpha_1 P = BP + (\beta_0 + \beta_1) PQ, \tag{2.27}$$

and

$$A + \alpha_0 P + \alpha_1 Q = BQ + (\beta_0 + \beta_1) PQ. \tag{2.28}$$

By subtracting, we have

$$(\alpha_0 - \alpha_1 + B)\ (P - Q) = 0. \tag{2.29}$$

This implies $P = Q$. This is a contradiction. Thus, equation (1.1) has no prime period two solution.

Case 3: σ and τ are odd positive integers. In this case, $x_{n+1} = x_{n-\sigma} = x_{n-\tau}$. From the difference equation (1.1), we have

$$P = \frac{A + \alpha_0 Q + \alpha_1 P}{B + \beta_0 Q + \beta_1 P}, \qquad Q = \frac{A + \alpha_0 P + \alpha_1 Q}{B + \beta_0 P + \beta_1 Q}.$$

Consequently, we obtain

$$A + \alpha_0 Q + \alpha_1 P = BP + \beta_0 PQ + \beta_1 P^2,$$

and

$$A + \alpha_0 P + \alpha_1 Q = BQ + \beta_0 PQ + \beta_1 Q^2.$$

By subtracting, we have

$$P + Q = -\frac{B + \overline{a}}{\beta_1}, \tag{2.30}$$

while, by adding we obtain

$$PQ = \frac{A\beta_1 - \alpha_0 (B + \overline{a})}{\overline{b}\beta_1}, \tag{2.31}$$

provided that $B + \overline{a} < 0$ and $\overline{b} > 0$. Assume that P and Q are two positive distinct real roots of the quadratic equation

$$t^2 - (\ P + Q)\, t + PQ = 0. \tag{2.32}$$

Thus, we deduce that

$$\left(-\frac{B + \overline{a}}{\beta_1}\right)^2 > 4\left(\frac{A\beta_1 - \alpha_0 (B + \overline{a})}{\overline{b}\beta_1}\right). \tag{2.33}$$

From (2.33), we obtain

$$4\beta_1 \left[A\beta_1 - \alpha_0 (B + \overline{a})\right] < \overline{b} (B + \overline{a})^2,$$

and hence, the condition (2.23) is valid. Conversely, suppose that the condition (2.23) is valid, provided that $B + \overline{a} < 0$ and $\overline{b} > 0$. Then, we deduce immediately from (2.23) that the inequality (2.23) holds. From which, there exist two positive distinct real numbers P and Q representing two positive roots of (2.32) such that

$$P = \frac{-\ (B+\overline{a})}{2\beta_1} - \frac{1}{2}\sqrt{T_1}, \tag{2.34}$$

and

$$Q = \frac{-\ (B+\overline{a})}{2\beta_1} + \frac{1}{2}\sqrt{T_1}, \tag{2.35}$$

where $T_1 > 0$ which is given by the formula

$$T_1 = \left(\frac{-\ (B+\overline{a})}{\beta_1}\right)^2 - 4\left(\frac{A\beta_1 - \alpha_0\,(B+\overline{a})}{\overline{b}\beta_1}\right). \tag{2.36}$$

Now, we are going to prove that P and Q are positive solutions of prime period two of the difference equation (1.1). To this end, we assume that $x_{-\omega} = Q$ and $x_0 = P$, where $\omega = \max\{\tau, \sigma\}$. Now, we are going to show that $x_1 = Q$ and $x_2 = P$. From the difference equation (1.1), we deduce that

$$x_1 = \frac{A + \alpha_0 P + \alpha_1 Q}{B + \beta_0 P + \beta_1 Q}. \tag{2.37}$$

By substituting (2.34)-(2.36) into (2.37), we obtain

$$x_1 = \frac{\frac{-2A\beta_1}{(B+\overline{a})} + \left[1+\sqrt{K_1}\right]\alpha_0 + \left[1-\sqrt{K_1}\right]\alpha_1}{\frac{-2B\beta_1}{(B+\overline{a})} + \left[1+\sqrt{K_1}\right]\beta_0 + \left[1-\sqrt{K_1}\right]\beta_1} = \frac{\left[\widetilde{a} - \frac{2A\beta_1}{(B+\overline{a})}\right] + \overline{a}\sqrt{K_1}}{\left[\widetilde{b} - \frac{2B\beta_1}{(B+\overline{a})}\right] + \overline{b}\sqrt{K_1}}, \tag{2.38}$$

where

$$K_1 = 1 - \left[\frac{4\beta_1\left[A\beta_1 - \alpha_0\,(B+\overline{a})\right]}{\overline{b}\ (B+\overline{a})^2}\right].$$

From the condition (2.23), we deduce that $K_1 > 0$. Multiplying the denominator and numerator of (2.38) by

$$\left(\widetilde{b} - \frac{2B\beta_1}{(B+\overline{a})}\right) - \overline{b}\sqrt{K_1},$$

we have

$$x_1 = \frac{\left[\widetilde{a} - \frac{2A\beta_1}{(B+\overline{a})}\right]\left[\widetilde{b} - \frac{2B\beta_1}{(B+\overline{a})}\right] - \overline{b}\overline{a}K_1}{\left[\widetilde{b} - \frac{2B\beta_1}{(B+\overline{a})}\right]^2 - \overline{b}^2 K_1} + \frac{\left[\widetilde{b}\overline{a} - \widetilde{a}\overline{b} - \overline{a}\frac{2B\beta_1}{(B+\overline{a})} + \overline{b}\frac{2A\beta_1}{(B+\overline{a})}\right]\sqrt{K_1}}{\left[\widetilde{b} - \frac{2B\beta_1}{(B+\overline{a})}\right]^2 - \overline{b}^2 K_1}.$$

After some reduction, we deduce that

$$x_1 = \frac{-\,(B+\ \overline{a})\left(1+\sqrt{K_1}\right)}{2\beta_1} = -\ \frac{B+\ \overline{a}}{2\beta_1} + \frac{1}{2}\sqrt{T_1} = Q.$$

Similarly, we can show that

$$x_2 = \frac{A + \alpha_0 x_1 + \alpha_1 x_{-(\omega-1)}}{B + \beta_0 x_1 + \beta_1 x_{-(\omega-1)}} = \frac{A + \alpha_0 Q + \alpha_1 P}{B + \beta_0 Q + \beta_1 P} = P.$$

By using induction, we have

$$x_n = Q \quad and \quad x_{n+1} = P \quad for \;\; all \;\; n \geq -\omega.$$

Thus, the difference eqution (1.1) has positive solutions of prime period two. Similarly, we can prove that if σ is a positive even integer and τ is a positive odd integer, then the necessary and sufficient condition for the difference equation (1.1) to have positive solutions of prime period two is that the condition (2.23) is valid, provided that $B + \overline{a} < 0$ and $\overline{b} > 0$. Thus, the proof of Theorem 6 is now complete. □

3. Numerical examples of the solutions of equation (1.1)

To illustrate the results of this paper, we consider numerical examples which represent different types of solutions to equation (1.1).

Example 1. Figure 1 shows that equation (1.1) has no prime period two solution if $\sigma = 4$, $\tau = 2$, $\omega = \max\{\tau, \sigma\} = 4$, $x_{-4} = 1$, $x_{-3} = 2$, $x_{-2} = 3$, $x_{-1} = 4$, $x_0 = 5$, $A = 40$, $B = 0.25$, $\alpha_0 = 3$, $\alpha_1 = 30$, $\beta_0 = 15$, $\beta_1 = 0.25$.

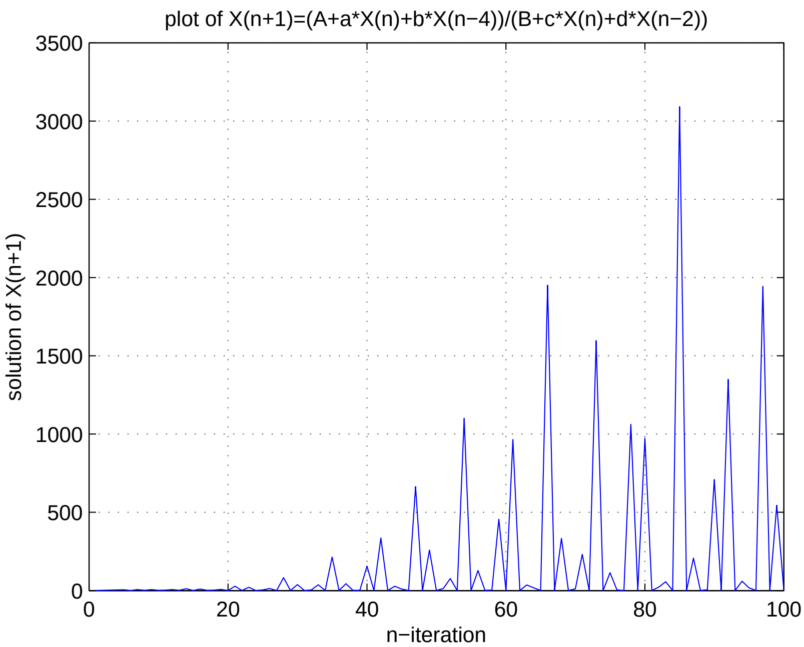

Figure 1. $\left(x_{n+1} = \dfrac{40 + 3x_n + 30x_{n-4}}{0.25 + 15x_n + 0.25x_{n-2}} \right)$

Example 2. Figure 2 shows that equation (1.1) has no prime period two solution if $\sigma = 2$, $\tau = 1$, $\omega = \max\{\tau, \sigma\} = 2$, $x_{-2} = 1$, $x_{-1} = 2$, $x_0 = 3$, $A = 40$, $B = 0.25$, $\alpha_0 = 3$, $\alpha_1 = 30$, $\beta_0 = 15$, $\beta_1 = 0$.

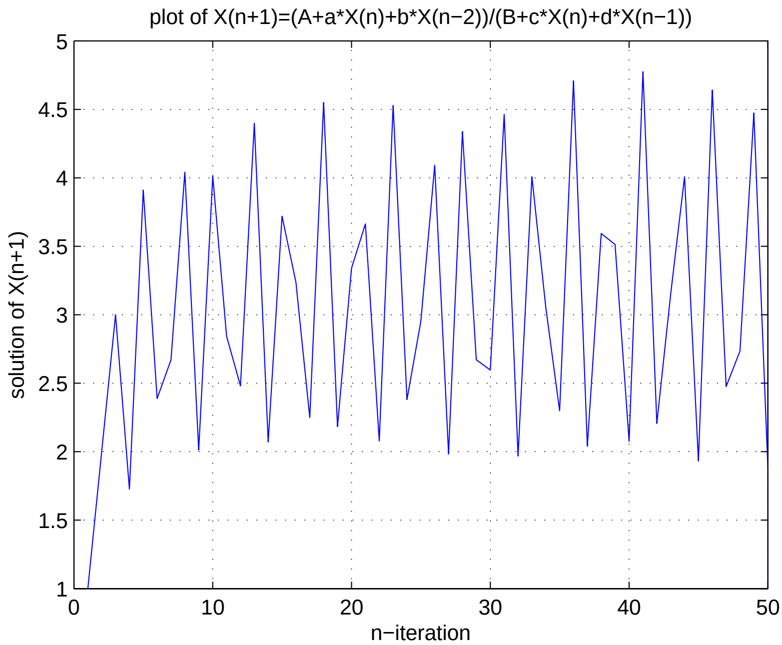

FIGURE 2. $\left(x_{n+1} = \dfrac{40 + 3x_n + 30x_{n-2}}{0.25 + 15x_n + 0.25x_{n-1}} \right)$

Example 3. Figure 3 shows that equation (1.1) has prime period two solution if $\sigma = 1$, $\tau = 3$, $\omega = \max\{\tau, \sigma\} = 3$, $x_{-3} = 35.2$, $x_{-2} = 71.8$, $x_{-1} = 35.2$, $x_0 = 71.8$, $A = 40$, $B = 0.25$, $\alpha_0 = 3$, $\alpha_1 = 30$, $\beta_0 = 15$, $\beta_1 = 0.25$.

Example 4. Figure 4 shows that the solution of equation (1.1) is global stability if $\sigma = 3$, $\tau = 2$, $\omega = \max\{\tau, \sigma\} = 3$, $x_{-3} = 1$, $x_{-2} = 2$, $x_{-1} = 3$, $x_0 = 4$, $A = 40$, $B = 100$, $\alpha_0 = 3$, $\alpha_1 = 30$, $\beta_0 = 10$, $\beta_1 = 0.5$.

The following example verifies the definitions 1-4 of Section 1:

Example 5. The following four special cases of equation (1.1):

$$x_{n+1} = \frac{1}{x_n}, \qquad n = 0, 1, 2, \ldots \tag{3.1}$$

$$x_{n+1} = \frac{1}{x_{n-1}}, \qquad n = 0, 1, 2, \ldots \tag{3.2}$$

$$x_{n+1} = \frac{1 + x_n}{x_{n-1}}, \qquad n = 0, 1, 2, \ldots \tag{3.3}$$

$$x_{n+1} = \frac{x_n}{x_{n-1}}, \qquad n = 0, 1, 2, \ldots \tag{3.4}$$

are remarkable in the following sense:

(i) The equilibrium points of equations (3.1)-(3.4) are $\widetilde{x} = \pm 1, \pm 1, \frac{1 \pm \sqrt{5}}{2}, 1$ respectively.

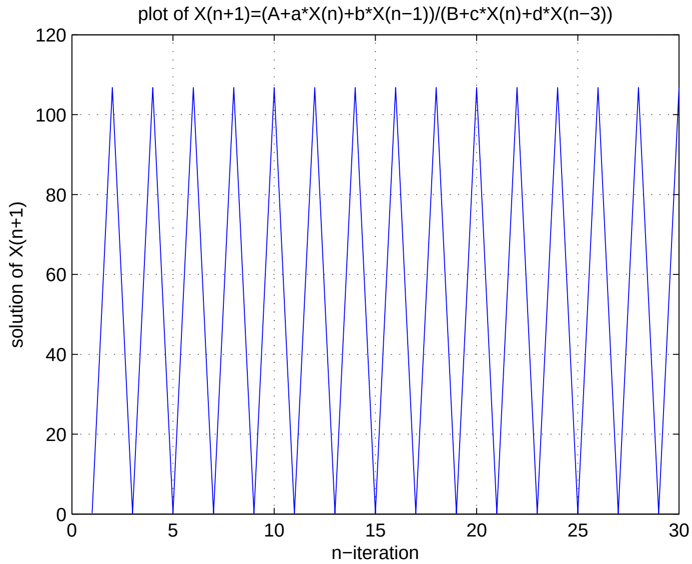

FIGURE 3. $\left(x_{n+1} = \dfrac{40 + 3x_n + 30x_{n-1}}{0.25 + 15x_n + 0.25x_{n-3}}\right)$

(ii) Every positive solution of equations (3.1)-(3.4) is periodic with period $= 2, 4, 5, 6$ respectively.

(iii) Suppose $x_0 = \alpha \in (0, \infty)$ is an initial value of equation (3.1) and let $|x_0 - \widetilde{x}| = |x_0 \mp 1| = |\alpha \mp 1| < \delta$ where $\delta > 0$. Then, we get

$$|x_n - \widetilde{x}| = |x_n \mp 1| = \left| \frac{1}{x_{n-1}} \mp 1 \right| = \frac{|x_{n-1} \mp 1|}{|x_{n-1}|}. \tag{3.5}$$

Consequently, we deduce from (3.5) that

$$\begin{aligned} |x_1 \mp 1| &= \frac{|x_0 \mp 1|}{x_0} < \frac{\delta}{\alpha}, \\ |x_2 \mp 1| &= \frac{|x_1 \mp 1|}{x_1} < \delta, \\ |x_3 \mp 1| &= \frac{|x_2 \mp 1|}{x_2} < \frac{\delta}{\alpha}, \end{aligned} \tag{3.6}$$

and so on. From (3.6), we deduce that

$$|x_n \mp 1| < \epsilon(\delta) \text{ for all } n \geq 0, \quad \epsilon(\delta) > 0. \tag{3.7}$$

From the inequality (3.7), we deduce the following properties:

(a) The equilibrium points $\widetilde{x} = \pm 1$ are locally stable.

(b) Since $|x_0 - \widetilde{x}| < \delta$ and from (3.7) we have $\lim_{n\to\infty} x_n = \pm 1$ then $\widetilde{x} = \pm 1$ are locally asymptotically stable.

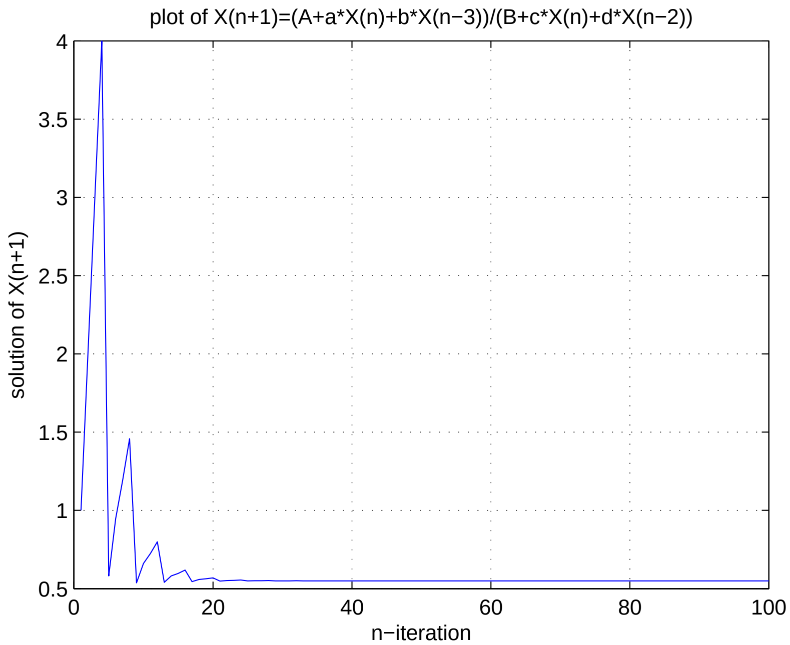

FIGURE 4. $\left(x_{n+1} = \dfrac{40 + 3x_n + 30x_{n-3}}{100 + 10x_n + 0.5x_{n-2}}\right)$

(c) Since $x_0 = \alpha \in (0, \infty)$ and from (3.7), we have $\lim_{n\to\infty} x_n = \pm 1$ then $\widetilde{x} = \pm 1$ are global attractor.

(d) From (a) and (c), we deduce that $\widetilde{x} = \pm 1$ are globally asymptotically stable.

(e) From (3.7), we have the inequality $\pm 1 - \epsilon(\delta) < x_n < \pm 1 + \epsilon(\delta)$. This implies that the sequence $\{x_n\}_{n=0}^{\infty}$ is bounded.

Similarly, we can investigate for the other equations (3.2)-(3.4) which are omitted here.

Remark 2. Examples 1, 2 verify Theorem 6 (i) which show that equation (1.1) has no prime period two solution, while Example 3 verifies Theorem 6 (ii) which shows that equation (1.1) has prime period two solution. But Example 4 verifies Theorem 5 which shows that the solution of equation (1.1) is globally asymptotically stable.

ACKNOWLEDGEMENTS

The authors wish to thank the referee for his suggestions and comments.

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