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A UNICITY THEOREM WITH TRUNCATED COUNTING FUNCTION FOR MEROMORPHIC MAPPINGS

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ABSTRACT. In this article, a unicity theorem with truncated multiplicities of meromorphic mappings in several complex variables sharing few targets are given. It gives some remarkable improvements for the results in [15].

1. Introduction

The unicity theorems with truncated multiplicities of meromorphic mappings of $\mathbb{C}^n$ into the complex projective space $\mathbb{P}^N(\mathbb{C})$ sharing a finite set of q fixed hyperplanes in $\mathbb{P}^N(\mathbb{C})$ have received much attention in the last few decades, and they are related to many problems in Nevanlinna theory and hyperbolic complex analysis (see the references in [1, 8, 14, 15, 16, 3, 4, 5] for the development in related subjects).

To state some of them, first of all we recall the following.

Let f be a nonconstant meromorphic mapping of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ and H a hyperplane in $\mathbb{P}^N(\mathbb{C})$ and k a positive integer or $k = \infty$. Denote by $\nu_{(f,H)}$ the map of $\mathbb{C}^n$ into $\mathbb{Z}$ whose value $\nu_{(f,H)}(a)$ $(a \in \mathbb{C}^n)$ is the intersection multiplicity of the images of f and H at $f(a)$.

For every $z \in \mathbb{C}^n$, we set

$$\nu_{(f,H),\le k}(z) = \begin{cases} 0 & \text{if } \nu_{(f,H)}(z) > k, \\ \nu_{(f,H)}(z) & \text{if } \nu_{(f,H)}(z) \le k, \end{cases}$$

$$\nu_{(f,H),>k}(z) = \begin{cases} \nu_{(f,H)}(z) & \text{if } \nu_{(f,H)}(z) > k, \\ 0 & \text{if } \nu_{(f,H)}(z) \le k. \end{cases}$$

We now take a linearly nondegenerate meromorphic mapping f of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$, a positive integer d and q hyperplanes $H_1, \dots, H_q$ in $\mathbb{P}^N(\mathbb{C})$ in general position with

$$\dim \{z \in \mathbb{C}^n : \nu_{(f,H_i),\le k}(z) > 0 \text{ and } \nu_{(f,H_j),\le k}(z) > 0\} \le n-2 \quad (1 \le i < j \le q).$$

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We consider the family $\mathcal{F}(f, \{H_j\}_{j=1}^q, k, d)$ of all meromorphic mappings $g : \mathbb{C}^n \to \mathbb{P}^N(\mathbb{C})$ satisfying the conditions

(a) g is linearly nondegenerate,
(b) $\min\{\nu_{(f,H_j),\leq k}, d\} = \min\{\nu_{(g,H_j),\leq k}, d\} \quad (1 \leq j \leq q)$,
(c) $f(z) = g(z)$ on $\bigcup_{j=1}^q \{z \in \mathbb{C}^n : \nu_{(f,H_j),\leq k}(z) > 0\}$.

Denote by $\sharp\, S$ the cardinality of the set S.

In [15], the authors showed that

Theorem 1. (see [15])

(1) *If $N = 1$, then $\sharp\, \mathcal{F}(f, \{H_i\}_{i=1}^{3N+1}, k, 2) \leq 2$ for $k \geq 15$.*
(2) *If $N \geq 2$, then $\sharp\, \mathcal{F}(f, \{H_i\}_{i=1}^{3N+1}, k, 2) \leq 2$ for $k \geq 3N + 3 + \frac{4}{N-1}$.*
(3) *If $N \geq 4$, then $\sharp\, \mathcal{F}(f, \{H_i\}_{i=1}^{3N}, k, 2) \leq 2$ for $k > 3N + 7 + \frac{24}{N-3}$.*
(4) *If $N \geq 6$, then $\sharp\, \mathcal{F}(f, \{H_i\}_{i=1}^{3N-1}, k, 2) \leq 2$ for $k > 3N + 11 + \frac{60}{N-5}$.*

We are going to improve Theorem 1. Namely, we prove the following

Theorem 2. *Let $f^1, f^2, f^3 : \mathbb{C}^n \longrightarrow \mathbb{P}^N(\mathbb{C})$ be three meromorphic mappings and let $\{H_i\}_{i=1}^q$ be hyperplanes in general position. Let $d, k, k_{1i}, k_{2i}, k_{3i}$ be the integers with $1 \leq k_{1i}, k_{2i}, k_{3i} \leq \infty$ $(1 \leq i \leq q)$. We set $M = \max\{k_{ji}\}$, $m = \min\{k_{ji}\}$ $(1 \leq j \leq 3, 1 \leq i \leq q)$, $k = \max\{\sharp\{i \in \{1, 2 \cdots, q\} \mid k_{ji} = m\} \mid 1 \leq j \leq 3\}$. Define $d = 0$ if $M = m$ and $d = \min\{k_{ji} - m > 0 \mid 1 \leq j \leq 3; 1 \leq i \leq q\}$ if $M \neq m$.*

Assume that the following conditions are satisfied

(a) $\dim\{z \in \mathbb{C}^n : \nu_{(f^j,H_i),\leq k_{ji}} > 0$ *and* $\nu_{(f^j,H_l),\leq k_{jl}} > 0\} \leq n - 2$
$(1 \leq j \leq 3; 1 \leq i < l \leq q)$,
(b) $\min(\nu_{(f^j,H_i),\leq k_{ji}}, 2) = \min(\nu_{(f^t,H_i),\leq k_{ti}}, 2)$
$(1 \leq j < t \leq 3; 1 \leq i \leq q)$,
(c) $f^1 \equiv f^j$ *on* $\bigcup_{\alpha=1}^q \{z \in \mathbb{C}^n : \nu_{(f^1,H_\alpha),\leq k_{1\alpha}}(z) > 0\} \quad (1 \leq j \leq 3)$.

Then $f^1 \equiv f^2$ or $f^2 \equiv f^3$ or $f^3 \equiv f^1$ if one of the following conditions is satisfied

(1) $N \geq 2, 3N - 1 \leq q \leq 3N + 1, m > 3N + 1 + \dfrac{16}{3(N-1)}$ *and*
$$(2q - 5N - 3) > \frac{2Nk}{m+1} + \frac{2N(q-k)}{m+d+1} - \frac{3N^2 + N}{M+1}.$$
(2) $N = 1, q = 4$ *and*
$$\frac{3(2k+1)}{m+1} + \frac{6(4-k)}{m+d+1} + \frac{6k}{M(m+1)} + \frac{24-6k}{M(m+d+1)} < 1 + \frac{12}{M}.$$

We now give some corollaries of Theorem 2.

***)** Theorem 1 is deduced immediately from Theorem 2 by choosing $M = m$ and $k = q$.

***)** When $k = 1, M = m + d$ and $d = 1$ or $d = 2$, by using the case 1 of Theorem 2, we have the following

Corollary 3. *Let $f^1, f^2, f^3 : \mathbb{C}^n \longrightarrow \mathbb{P}^N(\mathbb{C})$ be three meromorphic mappings and let $\{H_i\}_{i=1}^{3N+1}$ be hyperplanes in general position. Let k_i be the positive integers with $1 \le i \le 3N+1$ satisfying the following conditions*

(i) $\dim\{z \in \mathbb{C}^n : \nu_{(f^j,H_i),\le k_i} > 0$ *and* $\nu_{(f^j,H_l),\le k_l} > 0\} \le n-2 \quad (1 \le i < l \le 3N+1)$.
(ii) $\min(\nu_{(f^j,H_i),\le k_i}, 2) = \min(\nu_{(f^t,H_i),\le k_i}, 2) \quad (1 \le j < t \le 3; 1 \le i \le 3N+1)$.
(iii) $f^1 \equiv f^j$ *on* $\bigcup_{\alpha=1}^{3N+1}\{z \in \mathbb{C}^n : \nu_{(f^1,H_\alpha),\le k_\alpha}(z) > 0\} \quad (1 \le j \le 3)$.

Then $f^1 \equiv f^2$ or $f^2 \equiv f^3$ or $f^3 \equiv f^1$ if one of the following conditions is satisfied

(1) $N \ge 2, k_j = k_1 + 1$ *for every* $2 \le j \le 3N+1$ *and* $k_1 > 3N + 2 + \frac{14}{3(N-1)}$.
(2) $N \ge 2, k_j = k_1 + 2$ *for every* $2 \le j \le 3N+1$ *and* $k_1 > 3N + 1 + \frac{16}{3(N-1)}$.

***)** When $k = 1$ and $M = m + d$, by using the proof for the Case 2 of Theorem 2, we have the following

Corollary 4. *Let $f^1, f^2, f^3 : \mathbb{C}^n \longrightarrow \mathbb{P}^1(\mathbb{C})$ be three meromorphic functions and let $\{H_i\}_{i=1}^4$ be hyperplanes in general position. Let k_i $(1 \le i \le 4)$ be the positive integers satisfying the following conditions*

(i) $\dim\{z \in \mathbb{C}^n : \nu_{(f^j,H_i),\le k_i} > 0$ *and* $\nu_{(f^j,H_l),\le k_l} > 0\} \le n-2 \quad (1 \le j \le 3; 1 \le i < l \le 4)$,
(ii) $\min(\nu_{(f^j,H_i),\le k_i}, 2) = \min(\nu_{(f^t,H_i),\le k_i}, 2) \quad (1 \le j < t \le 3; 1 \le i \le 4)$; *and*
(iii) $f^1 \equiv f^j$ *on* $\bigcup_{\alpha=1}^{4}\{z \in \mathbb{C}^n : \nu_{(f^1,H_\alpha),\le k_\alpha}(z) > 0\} \quad (1 \le j \le 3)$.

Assume that one of the following conditions is satisfied

(1) $k_1 = 9, k_2 = k_3 = k_4 = 66$.
(2) $k_1 = 10, k_2 = k_3 = k_4 = 36$.
(3) $k_1 = 11, k_2 = k_3 = k_4 = 26$.
(4) $k_1 = 12, k_2 = k_3 = k_4 = 21$.
(5) $k_1 = 13, k_2 = k_3 = k_4 = 18$.
(6) $k_1 = 14, k_2 = k_3 = k_4 = 16$.

Then $f^1 \equiv f^2$ or $f^2 \equiv f^3$ or $f^3 \equiv f^1$.

2. Basic notions in Nevanlinna theory

2.1. We set $||z|| = \big(|z_1|^2 + \cdots + |z_n|^2\big)^{1/2}$ for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and define

$$B(r) := \{z \in \mathbb{C}^n : ||z|| < r\}, \quad S(r) := \{z \in \mathbb{C}^n : ||z|| = r\} \ (0 < r < \infty).$$

Define

$$v_{n-1}(z) := \big(dd^c||z||^2\big)^{n-1} \quad \text{and}$$

$$\sigma_n(z) := d^c\log||z||^2 \wedge \big(dd^c\log||z||^2\big)^{n-1} \text{on} \quad \mathbb{C}^n \setminus \{0\}.$$

2.2. Let F be a nonzero holomorphic function on a domain Ω in $\mathbb{C}^n$. For a multi-index $\alpha = (\alpha_1, ..., \alpha_n)$, we set $|\alpha| = \alpha_1 + ... + \alpha_n$ and $\mathcal{D}^\alpha F = \dfrac{\partial^{|\alpha|} F}{\partial^{\alpha_1} z_1 ... \partial^{\alpha_n} z_n}$. We define the mapping $\nu_F : \Omega \to \mathbb{Z}$ by

$$\nu_F(z) := \max \{m : \mathcal{D}^\alpha F(z) = 0 \text{ for all } \alpha \text{ with } |\alpha| < m\} \ (z \in \Omega).$$

We mean by a divisor on a domain Ω in $\mathbb{C}^n$ a mapping $\nu : \Omega \to \mathbb{Z}$ such that, for each $a \in \Omega$, there are nonzero holomorphic functions F and G on a connected neighborhood U of a $(\subset \Omega)$ such that $\nu(z) = \nu_F(z) - \nu_G(z)$ for each $z \in U$ outside an analytic set of dimension $\leq n-2$. Two divisors are regarded as the same if they are identical outside an analytic set of dimension $\leq n-2$. For a divisor ν on Ω we set $|\nu| := \overline{\{z : \nu(z) \neq 0\}}$, which is a purely $(n-1)$-dimensional analytic subset of Ω or empty.

Take a nonzero meromorphic function φ on a domain Ω in $\mathbb{C}^n$. For each $a \in \Omega$, we choose nonzero holomorphic functions F and G on a neighborhood $U \subset \Omega$ such that $\varphi = \dfrac{F}{G}$ on U and $\dim(F^{-1}(0) \cap G^{-1}(0)) \leq n-2$, and we define the divisors $\nu_\varphi, \ \nu_\varphi^\infty$ by $\nu_\varphi := \nu_F, \ \nu_\varphi^\infty := \nu_G$, which are independent of choices of F and G. Hence they are globally well-defined on Ω.

2.3. For a divisor ν on $\mathbb{C}^n$ and for positive integers k, M (or $M = \infty$), we define the counting functions of ν as follows. Set

$$\nu^{(M)}(z) = \min \{M, \nu(z)\},$$

$$\nu^{(M)}_{\leq k}(z) = \begin{cases} 0 & \text{if } \nu(z) > k, \\ \nu^{(M)}(z) & \text{if } \nu(z) \leq k, \end{cases}$$

$$\nu^{(M)}_{> k}(z) = \begin{cases} \nu^{(M)}(z) & \text{if } \nu(z) > k, \\ 0 & \text{if } \nu(z) \leq k. \end{cases}$$

We define $n(t)$ by

$$n(t) = \begin{cases} \int\limits_{|\nu| \cap B(t)} \nu(z) v_{n-1} & \text{if } n \geq 2, \\ \sum\limits_{|z| \leq t} \nu(z) & \text{if } n = 1. \end{cases}$$

Similarly, we define $n^{(M)}(t), \ n^{(M)}_{\leq k}(t), \ n^{(M)}_{>k}(t)$.

Define

$$N(r, \nu) = \int\limits_1^r \frac{n(t)}{t^{2n-1}} dt \quad (1 < r < \infty).$$

Similarly, we define $N(r, \nu^{(M)}), \ N(r, \nu^{(M)}_{\leq k}), \ N(r, \nu^{(M)}_{>k})$ and denote them by $N^{(M)}(r, \nu), \ N^{(M)}_{\leq k}(r, \nu), \ N^{(M)}_{>k}(r, \nu)$, respectively.

Let $\varphi : \mathbb{C}^n \longrightarrow \mathbb{C}$ be a meromorphic function. Define $N_\varphi(r) = N(r, \nu_\varphi)$, $N^{(M)}_\varphi(r) = N^{(M)}(r, \nu_\varphi)$, $N^{(M)}_{\varphi,\leq k}(r) = N^{(M)}_{\leq k}(r, \nu_\varphi)$, $N^{(M)}_{\varphi,>k}(r) = N^{(M)}_{>k}(r, \nu_\varphi)$.

For brevity we will omit the superscript $^{(M)}$ if $M = \infty$.

2.4. Let $f : \mathbb{C}^n \longrightarrow \mathbb{P}^N(\mathbb{C})$ be a meromorphic mapping. For arbitrarily fixed homogeneous coordinates $(w_0 : \cdots : w_N)$ on $\mathbb{P}^N(\mathbb{C})$, we take a reduced representation $f = (f_0 : \cdots : f_N)$, which means that each f_i is a holomorphic function on $\mathbb{C}^n$ and $f(z) = \big(f_0(z) : \cdots : f_N(z)\big)$ outside the analytic set $\{f_0 = \cdots = f_N = 0\}$ of codimension ≥ 2. Set $\|f\| = \big(|f_0|^2 + \cdots + |f_N|^2\big)^{1/2}$.

The characteristic function of f is defined by

$$T(r, f) = \int\limits_{S(r)} \log\|f\|\sigma_n - \int\limits_{S(1)} \log\|f\|\sigma_n.$$

Let H be a hyperplane in $\mathbb{P}^N(\mathbb{C})$ given by $H = \{a_0\omega_0 + ... + a_N\omega_N\}$, where $A := (a_0, ..., a_N) \neq (0, ..., 0)$. We set $(f, H) = \sum_{i=0}^N a_i f_i$. Then we can define the corresponding divisor $\nu_{(f,H)}$ which is rephrased as the intersection multiplicity of the image of f and H at $f(z)$. Moreover, we define the proximity function of H by

$$m_{f,H}(r) = \int_{S(r)} \log \frac{||f|| \cdot ||H||}{|(f, H)|}\sigma_n - \int_{S(1)} \log \frac{||f|| \cdot ||H||}{|(f, H)|}\sigma_n,$$

where $||H|| = (\sum_{i=0}^N |a_i|^2)^{\frac{1}{2}}$.

2.5. Let φ be a nonzero meromorphic function on $\mathbb{C}^n$, which are occasionally regarded as a meromorphic mapping into $\mathbb{P}^1(\mathbb{C})$. The proximity function of φ is defined by

$$m(r, \varphi) := \int_{S(r)} \log\max\ (|\varphi|, 1)\sigma_n.$$

2.6. As usual, by the notation $''||\ P''$ we mean the assertion P holds for all $r \in [0, \infty)$ excluding a Borel subset E of the interval $[0, \infty)$ with $\int_E dr < \infty$.

The following results play essential roles in Nevanlinna theory (see [11], [12], [13]).

First Main Theorem. *Let $f : \mathbb{C}^n \to \mathbb{P}^N(\mathbb{C})$ be a linearly nondegenerate meromorphic mapping and H be a hyperplane in $\mathbb{P}^N(\mathbb{C})$. Then*

$$N_{(f,H)}(r) + m_{f,H}(r) = T(r, f)\ (r > 1).$$

Second Main Theorem. *Let $f : \mathbb{C}^n \to \mathbb{P}^N(\mathbb{C})$ be a linearly nondegenerate meromorphic mapping and $H_1, \ldots, H_q$ be hyperplanes in general position in $\mathbb{P}^N(\mathbb{C})$. Then*

$$||\quad (q - N - 1)T(r, f) \leq \sum_{i=1}^q N^{(N)}_{(f,H_i)}(r) + o(T(r, f)).$$

Logarithmic Derivative Lemma. *Let f be a nonzero meromorphic function on $\mathbb{C}^n$. Then*

$$\Big|\Big| \quad m\left(r, \frac{\mathcal{D}^\alpha(f)}{f}\right) = O(\log^+ T(r,f)) \ (\alpha \in \mathbb{Z}^n_+).$$

3. Some auxiliary lemmas

Lemma 3.1. *Suppose $d \geq 1$ and $q \geq N+2$. Then*

$$|| \ T(r, f^\alpha) = O(T(r, f^1)) \text{ for each } (1 \leq \alpha \leq 3).$$

Proof. By the Second Main Theorem, we have

$$\begin{aligned} \Big|\Big| \quad (q-N-1)T(r,f^\alpha) &\leq \sum_{i=1}^{q} N^{(N)}_{(f^\alpha, H_i)}(r) + o(T(r,f^\alpha)) \\ &\leq \sum_{i=1}^{q} N \cdot N^{(1)}_{(f^\alpha, H_i)}(r) + o(T(r,f^\alpha)) \\ &= \sum_{i=1}^{q} N \cdot N^{(1)}_{(f^1, H_i)}(r) + o(T(r,f^\alpha)) \\ &\leq qNT(r,f^1) + o(T(r,f^\alpha)). \end{aligned}$$

Hence $|| \quad T(r, f^\alpha) = O(T(r, f^1))$.

Similarly, we get $||T(r, f^1) = O(T(r, f^\alpha))$. □

Take 3 mappings f^1, f^2, f^3 with reduced representations $f^k := (f^k_0 : \ldots : f^k_N)$ and set $T(r) := \sum_{k=1}^3 T(r, f^k)$. For each $c = (c_0, \ldots, c_N) \in \mathbb{C}^{N+1} \setminus \{0\}$, we define $(f^k, c) := \sum_{i=0}^N c_i f^k_i$ $(0 \leq k \leq N)$. Denote by $\mathcal{C}$ the set of all $c \in \mathbb{C}^{N+1} \setminus \{0\}$ such that

$$\dim\{z \in \mathbb{C}^n : (f^k, H_j)(z) = (f^k, c)(z) = 0\} \leq n-2.$$

Lemma 3.2. ([10, Lemma 5.1]) *$\mathcal{C}$ is dense in $\mathbb{C}^{N+1}$.*

Lemma 3.3. ([8]) *For every $c \in \mathcal{C}$, we put $F^{jk}_c = \dfrac{(f^k, H_j)}{(f^k, c)}$. Then*

$$T(r, F^{jk}_c) \leq T(r, f^k) + o(T(r)).$$

Definition 3.4. ([8]) Let $F_0, \ldots, F_M$ be meromorphic functions on $\mathbb{C}^n$, where $M \geq 1$. Take a set $\alpha := (\alpha^0, \ldots, \alpha^{M-1})$ whose components α^k are composed of n nonnegative integers, and set $|\alpha| = |\alpha^0| + \ldots + |\alpha^{M-1}|$. We define Cartan's auxiliary function by

$$\Phi^\alpha \equiv \Phi^\alpha(F_0, \ldots, F_M) := F_0 F_1 \cdots F_M \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \mathcal{D}^{\alpha^0}(\frac{1}{F_0}) & \mathcal{D}^{\alpha^0}(\frac{1}{F_1}) & \cdots & \mathcal{D}^{\alpha^0}(\frac{1}{F_M}) \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{D}^{\alpha^{M-1}}(\frac{1}{F_0}) & \mathcal{D}^{\alpha^{M-1}}(\frac{1}{F_1}) & \cdots & \mathcal{D}^{\alpha^{M-1}}(\frac{1}{F_M}) \end{vmatrix}$$

Proposition 3.5. ([7, Proposition 4.9]) *Let $\alpha=(\alpha^0,\cdots,\alpha^N)$ be an admissible set for $F=(f_0,\cdots,f_N)$ and let h be a holomorphic function. Then,*

$$\det\Big(D^{\alpha^0}(hF),\cdots,D^{\alpha^N}(hF)\Big)=h^{N+1}\det\Big(D^{\alpha^0}(F),\cdots,D^{\alpha^N}(F)\Big).$$

Lemma 3.6. ([8]) *If $\Phi^\alpha(F,G,H)=0$ and $\Phi^\alpha(\frac{1}{F},\frac{1}{G},\frac{1}{H})=0$ for all α with $|\alpha|\le 1$, then one of the following assertions holds:*

(i) *$F=G, G=H$ or $H=F$.*

(ii) *$\frac{F}{G},\frac{G}{H}$ and $\frac{H}{F}$ are all constant.*

Using the same argument in [8], we have both following lemmas

Lemma 3.7. *Suppose that $\Phi^\alpha(F_0,...,F_M)\not\equiv 0$ with $|\alpha|\le\dfrac{M(M-1)}{2}$. If*

$$\nu^{([d])}:=\min\{\nu_{F_0,\le k_0},d\}=\min\{\nu_{F_1,\le k_1},d\}=\cdots=\min\{\nu_{F_M,\le k_M},d\}$$

for some $d\ge|\alpha|$, then $\nu_{\Phi^\alpha}(z_0)\ge\min\{\nu^{([d])}(z_0),d-|\alpha|\}$ for every $z_0\in\{z:\nu_{F_0,\le k_0}(z)>0\}\setminus A$, where A is an analytic subset of codimension ≥ 2.

Proof. Set $H_s:=\{z:\nu_{F_s,\le k_s}(z)>0\}$, then by the assumption we have $H_0=H_1=...=H_M:=H$. Denote by A the set of all singularities of H. Then A is an analytic set of dimension at most $n-2$. We assume that $z_0\in H\setminus A$. We choose a nonzero holomorphic function h on a neighborhood U of z_0 such that dh has no zero and $H\cap U=\{z\in U;h(z)=0\}$. Set $m_s:=\nu_{F_s}(z_0)$ and $\varphi_s:=\dfrac{1}{F_s}$ for $0\le s\le M$. We can write $\varphi_s=h^{-m_s}\widetilde{\varphi}_s$ on a neighborhood $V(\subset U)$ of z_0, where $\widetilde{\varphi}_s$ are nowhere vanishing holomorphic functions on V.

We first consider the case $\nu^{[d]}(z_0)=d$. We have

$$\Phi^\alpha=\begin{vmatrix} F_0 & F_1 & \cdots & F_M\\ F_0.\mathcal{D}^{\alpha^0}(\frac{1}{F_0}) & F_1.\mathcal{D}^{\alpha^0}(\frac{1}{F_1}) & \cdots & F_M.\mathcal{D}^{\alpha^0}(\frac{1}{F_M})\\ \vdots & \vdots & \vdots & \vdots\\ F_0.\mathcal{D}^{\alpha^{M-1}}(\frac{1}{F_0}) & F_1.\mathcal{D}^{\alpha^{M-1}}(\frac{1}{F_1}) & \cdots & F_M.\mathcal{D}^{\alpha^{M-1}}(\frac{1}{F_M})\end{vmatrix}$$
$$=\sum_{i=0}^{M}(-1)^iF_i\psi_i,$$

where $\psi_i:=\det\left(\dfrac{D^{\alpha^l}\varphi_k}{\varphi_k};k=0,...,i-1,i+1,...,M;\ l=0,1,...,M-1\right)$ are meromorphic functions.

By induction on $|\alpha^l|$, we can write each $\dfrac{D^{\alpha^l}\varphi_k}{\varphi_k}$ as $\dfrac{D^{\alpha^l}\varphi_k}{\varphi_k}=\dfrac{\psi_{k,l}}{h^{|\alpha^l|}}$, where $\psi_{k,l}$ is a holomorphic function, and

$$\psi_i=\sum_{l=(l_1,...,l_M)}\epsilon(l)\frac{D^{\alpha^{l_1}}\varphi_0}{\varphi_0}...\frac{D^{\alpha^{l_i}}\varphi_{i-1}}{\varphi_{i-1}}.\frac{D^{\alpha^{l_{i+1}}}\varphi_{i+1}}{\varphi_{i+1}}...\frac{D^{\alpha^{l_M}}\varphi_M}{\varphi_M},$$

where $l = (l_1, ..., l_M)$ runs through all permutations of $\{0, 1, ..., M-1\}$ and $\epsilon(l)$ denotes the signature of a permutation l. This implies that $\nu^{\infty}_{\psi_i} \leq\mid \alpha \mid$. By the assumption $\nu_{F_i,\leq k_i}(z_0) \geq \nu^{[d]}(z_0) = d$, we have $\nu_{\Phi^\alpha}(z_0) \geq d- \mid \alpha \mid$.

After that, we consider the case $1 \leq \nu^{[d]}(z_0) < d$. Then, by the assumption, we get

$$m^* := m_0 = m_1 = \cdots = m_M = \nu^{[d]}(z_0).$$

We now write

$$\Phi^\alpha = \frac{1}{\varphi_0\varphi_1\cdots\varphi_M} \det\Big(D^{\alpha^l}(\varphi_k - \varphi_0); k = 1, ..., M;\ l = 0, 1, ..., M-1\Big),$$

and $\varphi_k - \varphi_0 = h^{-m^*}(\widetilde{\varphi}_k - \widetilde{\varphi}_0)$, where $\widetilde{\varphi}_k - \widetilde{\varphi}_0$ is a holomorphic function.

By applying Proposition 3.5, it follows that

$$\Phi^\alpha = \frac{h^{m^*(M+1)}}{\widetilde{\varphi}_0\widetilde{\varphi}_1...\widetilde{\varphi}_M}.\frac{1}{h^{m^*M}} \det\Big(D^{\alpha^l}(\widetilde{\varphi}_k - \widetilde{\varphi}_0); k = 1, ..., M;\ l = 0, 1, ..., M-1\Big),$$

and hence

$$\Phi^\alpha = \frac{h^{m^*}}{\widetilde{\varphi}_0\widetilde{\varphi}_1...\widetilde{\varphi}_M} \det\Big(D^{\alpha^l}(\widetilde{\varphi}_k - \widetilde{\varphi}_0); k = 1, ..., M;\ l = 0, 1, ..., M-1\Big).$$

This yields that $\nu_{\Phi^\alpha}(z_0) \geq m^*$. The proof is complete. □

Lemma 3.8. *Suppose that the assumptions in Lemma 3.7 are satisfied. If* $F_0 = \cdots = F_M \not\equiv 0, \infty$ *on an analytic subset* H *of pure dimension* $n-1$, *then* $\nu_{\Phi^\alpha}(z_0) \geq M,\ \forall\ z_0 \in H$.

Lemma 3.9. *Let* $f : \mathbb{C}^n \to \mathbb{P}^N(\mathbb{C})$ *be a linearly nondegenerate meromorphic mapping. Let* $H_1, H_2, \ldots, H_q$ *be* q *hyperplanes in* $\mathbb{P}^N(\mathbb{C})$ *located in general position. Assume that* $k_j \geq N-1$ $(1 \leq j \leq q)$. *Then*

$$\Big|\Big|\Big(q - N - 1 - \sum_{j=1}^{q} \frac{N}{k_j+1}\Big)T(r,f) \leq \sum_{j=1}^{q}\Big(1 - \frac{N}{k_j+1}\Big)N^{(N)}_{(f,H_j),\leq k_j}(r) + o(T(r,f)).$$

Proof. By the Second Main Theorem, we have

$$\begin{aligned}
&||\quad (q-N-1)T(r,f)\\
&\leq \sum_{j=1}^{q} N^{(N)}_{(f,H_j)}(r) + o(T(r,f))\\
&= \sum_{j=1}^{q} N^{(N)}_{(f,H_j),\leq k_j}(r) + \sum_{j=1}^{q} N^{(N)}_{(f,H_j),>k_j}(r) + o(T(r,f))\\
&\leq \sum_{j=1}^{q} N^{(N)}_{(f,H_j),\leq k_j}(r) + \sum_{j=1}^{q} \frac{N}{k_j+1} N_{(f,H_j),>k_j}(r) + o(T(r,f))\\
&= \sum_{j=1}^{q} N^{(N)}_{(f,H_j),\leq k_j}(r) + \sum_{j=1}^{q} \frac{N}{k_j+1}\Big(N_{(f,H_j)}(r) - N_{(f,H_j),\leq k_j}(r)\Big) + o(T(r,f))
\end{aligned}$$

$$\leq \sum_{j=1}^{q} \big(1 - \frac{N}{k_j+1}\big) N^{(N)}_{(f,H_j),\leq k_j}(r) + \sum_{j=1}^{q} \frac{N}{k_j+1} T(r,f) + o(T(r,f)).$$

Thus we have a desired inequality. □

Lemma 3.10. *Assume that there exists $\Phi^\alpha = \Phi^\alpha(F_c^{j_0 0}, \ldots, F_c^{j_0 M}) \not\equiv 0$ for some $c \in \mathcal{C}$, $|\alpha| \leq \dfrac{M(M-1)}{2}$, $2 \geq |\alpha|$ and the assumptions in Lemma 3.7 are satisfied. Then, for each $0 \leq i \leq M$, the following holds:*

$$\begin{aligned} || \ & N^{(2-|\alpha|)}_{(f^i,H_{j_0}),\leq k_{ij_0}}(r) + M \sum_{j\neq j_0} N^{(1)}_{(f^i,H_j),\leq k_{ij}}(r) \\ \leq & N(r,\nu_{\Phi^\alpha}) \\ \leq & T(r) + \sum_{l=0}^{M} N^{(\frac{M(M-1)}{2})}_{(f^l,H_{j_0}),>k_{lj_0}}(r) + o(T(r)). \end{aligned}$$

Proof. The first inequality is deduced immediately from Lemmas 3.7 and 3.8. On the other hand, we also have

$$N(r,\nu_{\Phi^\alpha}) \leq T(r,\Phi^\alpha) + O(1) = N(r,\nu^\infty_{\Phi^\alpha}) + m(r,\Phi^\alpha) + O(1). \tag{3.1}$$

We easily see that a pole of Φ^α is a zero or a pole of some $F_c^{j_0 l}$ and Φ^α is holomorphic at all zeros with multiplicities $\leq k_{lj_0}$ of $F_c^{j_0 l}$ because of Lemma 3.7 ($l \in \{0,\ldots,M\}$). Assume that z_0 is a zero of $F_c^{j_0 l}$ with multiplicity $> k_{lj_0}$. We also see that if z_0 is a pole of $\dfrac{\mathcal{D}^{\alpha_i}(1/F_c^{j_0 l})}{(1/F_c^{j_0 l})}$, then it has the multiplicity $\leq |\alpha_i|$. Thus, if z_0 is a pole of Φ^α then it has the multiplicity $\leq |\alpha| = \sum_{i=0}^{M-1} |\alpha_i| \leq \frac{M(M-1)}{2}$. This implies that

$$N(r,\nu^\infty_{\Phi^\alpha}) \leq \sum_{i=0}^{M} N^{(\frac{M(M-1)}{2})}_{(f^i,H_{j_0}),>k_{ij_0}}(r) + \sum_{i=0}^{M} N(r,\nu^\infty_{F_c^{j_0 i}}) \tag{3.2}$$

and

$$\begin{aligned} m(r,\Phi^\alpha) &\leq \sum_{i=0}^{M} m(r,F_c^{j_0 i}) + O\bigg(\sum m\bigg(r, \frac{\mathcal{D}^{\alpha_i}(\varphi_c^{j_0 k})}{\varphi_c^{j_0 k}}\bigg)\bigg) + O(1) \\ &\leq \sum_{i=0}^{M} m(r,F_c^{j_0 i}) + o(T(r)), \end{aligned} \tag{3.3}$$

where $\varphi_c^{j_0 k} = 1/F_c^{j_0 k}$. By (3.1), (3.2) and (3.3), we get

$$\begin{aligned} N(r,\nu_{\Phi^\alpha}) &\leq \sum_{i=0}^{M} N^{(\frac{M(M-1)}{2})}_{(f^i,H_{j_0}),>k_{ij_0}}(r) + \sum_{i=0}^{M} T(r,F_c^{j_0 i}) + o(T(r)) \\ &\leq T(r) + \sum_{i=0}^{M} N^{(\frac{M(M-1)}{2})}_{(f^i,H_{j_0}),>k_{ij_0}}(r) + o(T(r)). \end{aligned}$$

□

4. Proof of Theorem 2

Case 1. $N \geq 2, 3N-1 \leq q \leq 3N+1, m > 3N+1+\dfrac{16}{3(N-1)}$ and

$$(2q-5N-3) > \frac{2Nk}{m+1} + \frac{2N(q-k)}{m+d+1} - \frac{3N^2+N}{M+1}.$$

First, we need the following

Claim 1. Denote by $\mathcal{Q}$ the set of all indices $j_0 \in \{1, 2, \ldots, q\}$ satisfying the following: There exist $c \in \mathcal{C}$ and $\alpha = (\alpha_0, \alpha_1)$ with $|\alpha| \leq 1$ such that

$$\Phi^\alpha(F_c^{j_01}, F_c^{j_02}, F_c^{j_03}) \not\equiv 0.$$

Then $\mathcal{Q}$ is an empty set.

Proof. Assume that $\mathcal{Q}$ is non-empty. For every $1 \leq i \leq 3$ and $j_0 \in \mathcal{Q}$, by Lemma 3.10, we have

$$\begin{aligned} \Big\| \quad & N^{(1)}_{(f^i,H_{j_0}),\leq k_{ij_0}}(r) + 2\sum_{j\neq j_0} N^{(1)}_{(f^i,H_j),\leq k_{ij}}(r) \\ \leq & T(r) + \sum_{l=1}^{3} N^{(1)}_{(f^l,H_{j_0}),>k_{lj_0}}(r) + o(T(r)), \end{aligned}$$

and hence

$$\begin{aligned} \Big\| \quad & N^{(N)}_{(f^i,H_{j_0}),\leq k_{ij_0}}(r) + 2\sum_{j\neq j_0} N^{(N)}_{(f^i,H_j),\leq k_{ij}}(r) \\ \leq & N\cdot T(r) + N\sum_{l=1}^{3} N^{(1)}_{(f^l,H_{j_0}),>k_{lj_0}}(r) + o(T(r)). \end{aligned}$$

This implies that

$$\begin{aligned} \Big\| \quad & \sum_{i=1}^{3}\Big(N^{(N)}_{(f^i,H_{j_0}),\leq k_{ij_0}}(r) + 2\sum_{j\neq j_0} N^{(N)}_{(f^i,H_j),\leq k_{ij}}(r)\Big) \\ \leq & 3NT(r) + 3N\sum_{i=1}^{3} N^{(1)}_{(f^i,H_{j_0}),>k_{ij_0}}(r) + o(T(r)) \\ \leq & 3NT(r) + \sum_{i=1}^{3}\Big(\frac{3N}{k_{ij_0}+1}\Big) N_{(f^i,H_{j_0}),>k_{ij_0}}(r) + o(T(r)) \\ \leq & 3NT(r) + \sum_{i=1}^{3}\Big(\frac{3N}{k_{ij_0}+1}\Big)\Big(N_{(f^i,H_{j_0})}(r) - N_{(f^i,H_{j_0}),\leq k_{ij_0}}(r)\Big) + o(T(r)). \end{aligned} \tag{4.1}$$

Hence we see

$$\Big\|\ \sum_{i=1}^{3}\Big(2\sum_{j=1}^{q}N^{(N)}_{(f^i,H_j),\le k_{ij}}(r)\Big)$$

$$\le 3NT(r)+\sum_{i=1}^{3}\Big(\frac{3N}{k_{ij_0}+1}\Big)N_{(f^i,H_{j_0})}(r)$$

$$+\sum_{i=1}^{3}(1-\frac{3N}{k_{ij_0}+1})N^{(N)}_{(f^i,H_{j_0}),\le k_{ij_0}}(r)+o(T(r)). \tag{4.2}$$

On the other hand, since $1-\dfrac{3N}{k_{ij_0}+1}>0$ and

$$\max\{N^{(N)}_{(f^i,H_{j_0}),\le k_{ij_0}}(r);N_{(f^i,H_{j_0})}(r)\}\le T(r,f^i)+o(T(r,f^i)),\quad \forall 1\le i\le 3, \tag{4.3}$$

we have

$$\Big\|\quad 2\sum_{i=1}^{3}\sum_{j=1}^{q}N^{(N)}_{(f^i,H_j),\le k_{ij}}(r)\le (3N+1)T(r)+o(T(r)). \tag{4.4}$$

Using Lemma 3.9, we have

$$\Big\|\Big(q-N-1-\sum_{j=1}^{q}\frac{N}{k_{ij}+1}\Big)T(r,f^i)$$

$$\le\sum_{j=1}^{q}\Big(1-\frac{N}{k_{ij}+1}\Big)N^{(N)}_{(f^i,H_j),\le k_{ij}}(r)+o(T(r,f^i)).$$

$$\Rightarrow\qquad \Big(q-N-1-\frac{Nk}{m+1}-\frac{N(q-k)}{m+d+1}\Big)T(r,f^i)$$

$$\le\Big(1-\frac{N}{M+1}\Big)\sum_{j=1}^{q}N^{(N)}_{(f^i,H_j),\le k_{ij}}(r)+o(T(r,f^i)).$$

$$\Rightarrow\qquad \Big(q-N-1-\frac{Nk}{m+1}-\frac{N(q-k)}{m+d+1}\Big)T(r)$$

$$\le\Big(1-\frac{N}{M+1}\Big)\sum_{i=1}^{3}\sum_{j=1}^{q}N^{(N)}_{(f^i,H_j),\le k_{ij}}(r)+o(T(r)). \tag{4.5}$$

From (4.4) and (4.5), we have

$$\Big\|\ 2\Big(q-N-1-\frac{Nk}{m+1}-\frac{N(q-k)}{m+d+1}\Big)T(r)$$

$$\le(3N+1)(1-\frac{N}{M+1})T(r)+o(T(r)).$$

Letting $r \to +\infty$, we get

$$\Big\|\ 2\Big(q - N - 1 - \frac{Nk}{m+1} - \frac{N(q-k)}{m+d+1}\Big) \le (3N+1)\left(1 - \frac{N}{M+1}\right)$$

and hence

$$(2q - 5N - 3) \le \frac{2Nk}{m+1} + \frac{2N(q-k)}{m+d+1} - \frac{3N^2+N}{M+1}. \tag{4.6}$$

This is a contradiction. So we have $\sharp\mathcal{Q} = 0$. □

Claim 2. If $\sharp\Big(\{1, 2, \ldots, q\} \setminus \mathcal{Q}\Big) \ge 3N - 1$ and $N \ge 2$ then $f^1 \equiv f^2$, or $f^2 \equiv f^3$, or $f^3 \equiv f^1$.

Proof. Indeed, assume that $1, \ldots, 3N-1 \notin \mathcal{Q}$. By the density of $\mathcal{C}$, it follows that

$$\Phi^\alpha(F_j^{i1}, F_j^{i2}, F_j^{i3}) = 0 \ (1 \le i, j \le 3N-1, |\alpha| \le 1).$$

Thus, there exists $\chi_{ij} \ne 0$ such that $F_j^{i1} = \chi_{ij} F_j^{i2}$, or $F_j^{i2} = \chi_{ij} F_j^{i3}$ or $F_j^{i3} = \chi_{ij} F_j^{i1}$. We may assume that $F_j^{i1} = \chi_{ij} F_j^{i2}$.

Suppose $\chi_{ij} \ne 1$. Then we have the following: If $\nu_{(f^1,H_l),\le k_{1l}}(z) > 0$ $(l \ne i, j)$, then $\nu_{(f^1,H_i)}(z) > 0$ or $\nu_{(f^1,H_j)}(z) > 0$.

So we get $\sum_{l\ne i,j} \nu^{(1)}_{(f^1,H_l),\le k_{1l}}(z) \le \nu^{(1)}_{(f^1,H_i),>k_{1i}}(z) + \nu^{(1)}_{(f^1,H_j),>k_{1j}}(z)$ outside a finite union of analytic sets of dimension $\le n-2$. Hence

$$\begin{aligned}\sum_{l\ne i,j} N^{(1)}_{(f^1,H_l),\le k_{1l}}(r) &\le N^{(1)}_{(f^1,H_i),>k_{1i}}(r) + N^{(1)}_{(f^1,H_j),>k_{1j}}(r)\\ &\le \frac{1}{k_{1i}+1} N_{(f^1,H_i),>k_{1i}}(r) + \frac{1}{k_{1j}+1} N_{(f^1,H_j),>k_{1j}}(r)\\ &\le \frac{1}{k_{1i}+1} N_{(f^1,H_i)}(r) + \frac{1}{k_{1j}+1} N_{(f^1,H_j)}(r)\\ &\le \frac{2}{m+1} T(r, f^1).\end{aligned}$$

By Lemma 3.9 and since $k_{1l} \ge N - 1$, we have

$$\begin{aligned}&\Big\|\ \Big(q - N - 3 - \sum_{l\ne i,j} \frac{N}{k_{1l}+1}\Big) T(r, f^1)\\ &\le \sum_{l\ne i,j} \Big(1 - \frac{N}{k_{1l}+1}\Big) N^{(N)}_{(f^1,H_l),\le k_{1l}}(r) + o(T(r, f^1)).\end{aligned}$$

This yields that

$$\begin{aligned}&\Big(q - N - 3 - \sum_{l\ne i,j} \frac{N}{m+1}\Big) T(r, f^1)\\ &\le \sum_{l\ne i,j} \Big(1 - \frac{N}{M+1}\Big) N^{(N)}_{(f^1,H_l),\le k_{1l}}(r) + o(T(r, f^1))\end{aligned}$$

$$\le N\left(1-\frac{N}{M+1}\right)\sum_{l\ne i,j} N^{(1)}_{(f^1,H_l),\le k_{1l}}(r)+o(T(r,f^1))$$
$$\le\left(1-\frac{N}{M+1}\right)\frac{2N}{m+1}T(r,f^1)+o(T(r,f^1)).$$

Hence

$$\left(q-N-3-\frac{N(q-2)}{m+1}\right)\le\left(1-\frac{N}{M+1}\right)\frac{2N}{m+1}.$$

This means that

$$q-N-3-\frac{N(q-2)}{m+1}\le\frac{2N}{m+1}-\frac{2N^2}{(m+1)(M+1)}.$$

Thus

$$q-N-3\le\frac{Nq}{m+1}-\frac{2N^2}{(m+1)(M+1)}. \tag{4.7}$$

Moreover, since $N\ge 2,\ 3N+1\ge q$ and $m>3N+1+\dfrac{16}{3(N-1)}$, we have

$$\frac{(3N-3)}{2}\ge\frac{Nq}{m+1}$$

and

$$\frac{Nk}{m+1}+\frac{N(q-k)}{m+d+1}\ge\frac{Nq}{m+d+1}\ge\frac{Nq}{M+1}\ge\frac{3N^2+N}{2(M+1)}.$$

This implies that

$$\frac{5N+3}{2}+\frac{Nk}{m+1}+\frac{N(q-k)}{m+d+1}-\frac{3N^2+N}{2(M+1)}$$
$$>N+3+\frac{Nq}{m+1}-\frac{2N^2}{(m+1)(M+1)}.$$

Combining the hypothesis and (4.7), we get a contradiction. Hence $\chi_{ij}=1$.

We define the subsets I_1, I_2 and I_3 by

$I_1=\{i:\ 1\le i\le 3N-2 \text{ and } F^{i1}_{3N-1}=F^{i2}_{3N-1}\}$,

$I_2=\{i:\ 1\le i\le 3N-2 \text{ and } F^{i2}_{3N-1}=F^{i3}_{3N-1}\}$,

$I_3=\{i:\ 1\le i\le 3N-2 \text{ and } F^{i3}_{3N-1}=F^{i1}_{3N-1}\}$.

Then one of them contains at least N indices. We may assume that $\sharp I_1\ge N$. Then $f^1\equiv f^2$. Thus the claim is proved. □

From Claim 1 and Claim 2 and $q\ge 3N-1$, Case 1 is proved.

Case 2. Assume that $N=1$ and $q=4$.

For each $j_0\in\mathcal{Q}$, from (4.1), we get

$$\Big\|\ \sum_{i=1}^{3}\left(2\sum_{j=1}^{q}N^{(1)}_{(f^i,H_j),\le k_{ij}}(r)\right)$$

$$\leq 3T(r)+\sum_{i=1}^{3}\left(\frac{3}{k_{ij_0}+1}\right)(N_{(f^i,H_{j_0})}(r)-N^{(1)}_{(f^i,H_{j_0}),\leq k_{ij_0}}(r))$$
$$+\sum_{i=1}^{3}N^{(1)}_{(f^i,H_{j_0}),\leq k_{ij_0}}(r)+o(T(r))$$

and $N^{(1)}_{(f^i,H_{j_0}),\leq k_{ij_0}}(r)\leq N_{(f^i,H_{j_0})}(r)\leq T(r,f^i)+o(T(r))$ $(1\leq i\leq 3)$.

Hence

$$\Big\|2\sum_{i=1}^{3}\sum_{j=1}^{4}N^{(1)}_{(f^i,H_j),\leq k_{ij}}(r)$$
$$\leq 3(1+\frac{1}{m_{j_0}+1})T(r)+\sum_{i=1}^{3}(1-\frac{3}{m_{j_0}+1})N^{(1)}_{(f^i,H_{j_0}),\leq k_{ij_0}}(r)+o(T(r))$$
$$\leq 3(1+\frac{1}{m_{j_0}+1})T(r)+\sum_{i=1}^{3}(1-\frac{3}{m_{j_0}+1})N^{(1)}_{(f^i,H_{j_0}),\leq k_{ij_0}}(r)+o(T(r)), \tag{4.8}$$

where $m_j=min\{k_{ij}\mid 1\leq i\leq 3\}(1\leq j\leq 4)$.

On the other hand, from Lemma 3.9, we have

$$\Big\|\left(2-\sum_{j=1}^{4}\frac{1}{k_{ij}+1}\right)T(r,f^i)\leq\sum_{j=1}^{4}\left(1-\frac{1}{k_{ij}+1}\right)N^{(1)}_{(f^i,H_j),\leq k_{ij}}(r)+o(T(r,f^i)).$$

This implies that

$$\left(2-\frac{k}{m+1}-\frac{4-k}{m+d+1}\right)T(r,f^i)\leq\sum_{j=1}^{4}\left(1-\frac{1}{M+1}\right)N^{(1)}_{(f^i,H_j),\leq k_{ij}}(r)+o(T(r,f^i)).$$

Hence

$$\left(2-\frac{k}{m+1}-\frac{4-k}{m+d+1}\right)T(r)\leq\sum_{i=1}^{3}\sum_{j=1}^{4}\left(1-\frac{1}{M+1}\right)N^{(1)}_{(f^i,H_j),\leq k_{ij}}(r)+o(T(r)). \tag{4.9}$$

From (4.8) and (4.9), we have

$$\Big\|2\left(2-\frac{k}{m+1}-\frac{4-k}{m+d+1}\right)(\frac{M+1}{M})T(r)$$
$$\leq 3(1+\frac{1}{m_{j_0}+1})T(r)+\sum_{i=1}^{3}(1-\frac{3}{m_{j_0}+1})N^{(1)}_{(f^i,H_{j_0}),\leq k_{ij_0}}(r)+o(T(r)).$$

This yields that

$$\sum_{i=1}^{3}N^{(1)}_{(f^i,H_{j_0}),\leq k_{ij_0}}(r)\geq\left(\frac{m_{j_0}+1}{m_{j_0}-2}\right)\Bigg(2(2-\frac{k}{m+1}-\frac{4-k}{m+d+1})(\frac{M+1}{M})$$

$$- 3(1 + \frac{1}{m_{j_0}+1})\Big)T(r) + o(T(r)).$$

Hence

$$\sum_{i=1}^{3} N^{(1)}_{(f^i,H_{j_0}),\le k_{ij_0}}(r) \ge \left(\frac{m_{j_0}+1}{m_{j_0}-2}\right)\Big(2(2-\frac{k}{m+1}-\frac{4-k}{m+d+1})(\frac{M+1}{M}) - 3(1+\frac{1}{m_{j_0}+1})\Big)T(r) + o(T(r)). \tag{4.10}$$

Assume that $\sharp\mathcal{Q} \ge 3$, i.e, $\mathcal{Q} \supset \{j_0, j_1, j_2\}$. By (4.10), we get

$$\Big|\Big| \sum_{i=1}^{3}\sum_{s=0}^{2} N^{(1)}_{(f^i,H_{j_s}),\le k_{ij_s}}(r) \ge \sum_{s=0}^{2}\left(\frac{m_{j_s}+1}{m_{j_s}-2}\right)\Big(2(2-\frac{k}{m+1}-\frac{4-k}{m+d+1})(\frac{M+1}{M}) - 3(1+\frac{1}{m_{j_s}+1})\Big)T(r) + o(T(r)). \tag{4.11}$$

Since there exists $c \in \mathcal{C}$ such that $F_c^{j_0 1} - F_c^{j_0 2} \not\equiv 0$, it follows that

$$\sum_{s=0}^{2} N^{(1)}_{(f^i,H_{j_s}),\le k_{ij_s}}(r) \le N_{F_c^{j_0 1}-F_c^{j_0 2}}(r) \le T(r,f^1) + T(r,f^2) + O(1).$$

Similarly, we have

$$\sum_{s=0}^{2} N^{(1)}_{(f^i,H_{j_s}),\le k_{ij_s}}(r) \le T(r,f^2) + T(r,f^3) + O(1)$$

and

$$\sum_{s=0}^{2} N^{(1)}_{(f^i,H_{j_s}),\le k_{ij_s}}(r) \le T(r,f^3) + T(r,f^1) + O(1).$$

Hence

$$\sum_{s=0}^{2} N^{(1)}_{(f^i,H_{j_s}),\le k_{ij_s}}(r) \le \frac{2}{3}\cdot T(r) + O(1) \ (1 \le i \le 3)$$

and

$$\sum_{i=1}^{3}\sum_{s=0}^{2} N^{(1)}_{(f^i,H_{j_s}),\le k_{ij_s}}(r) \le 2.T(r) + O(1). \tag{4.12}$$

From (4.11) and (4.12), we have

$$2.T(r) \ge \sum_{s=0}^{2}\left(\frac{m_{j_s}+1}{m_{j_s}-2}\right)\Big(2(2-\frac{k}{m+1}-\frac{4-k}{m+d+1})(\frac{M+1}{M}) - 3(1+\frac{1}{m_{j_s}+1})\Big)T(r) + o(T(r)).$$

Letting $r \to +\infty$, we get

$$2 \geq \sum_{s=0}^{2} \left(\frac{m_{j_s}+1}{m_{j_s}-2}\right)\left(2(2-\frac{k}{m+1}-\frac{4-k}{m+d+1})(\frac{M+1}{M})-3(1+\frac{1}{m_{j_s}+1})\right).$$

On the other hand, the following function is increasing for $t > 2$

$$f(t) = \left(\frac{t+1}{t-2}\right)\left(2(2-\frac{k}{m+1}-\frac{4-k}{m+d+1})(\frac{M+1}{M})-3(1+\frac{1}{t+1})\right).$$

So we get

$$2 \geq 3.\left(\frac{m+1}{m-2}\right)\left(2(2-\frac{k}{m+1}-\frac{4-k}{m+d+1})(\frac{M+1}{M})-3(1+\frac{1}{m+1})\right).$$

This means that

$$\frac{2(m-2)}{3(m+1)} \geq \left(2(2-\frac{k}{m+1}-\frac{4-k}{m+d+1})(\frac{M+1}{M})-3(1+\frac{1}{m+1})\right).$$

Thus, we get

$$\frac{3(2k+1)}{m+1}+\frac{6(4-k)}{m+d+1}+\frac{6k}{M(m+1)}+\frac{24-6k}{M(m+d+1)} \geq 1+\frac{12}{M}.$$

This is a contradiction (Remarking that the equality does not happen if $\max_{1\leq j\leq 4}\{m_j\} > m$). Hence $\sharp\mathcal{Q} \leq 2$.

We now use the same argument in [15] to complete Case 2.

Without loss of generality, we may assume that $1, 2 \notin \mathcal{Q}$. By the density of $\mathcal{C}$ in $\mathbb{C}^2$, it follows that $\Phi^\alpha(F_j^{i0}, F_j^{i1}, F_j^{i2}) = 0$ for each $1 \leq i \leq 2, 1 \leq j \leq 2$ and for each $\alpha = (\alpha_0, \alpha_1)$ with $|\alpha| \leq 1$, where $F_j^{ik} = \dfrac{(f^k, H_i)}{(f^k, H_j)}$.

Applying Lemma 3.6 for $i = 1, j = 2$, we have the following two cases.

(i) There exist $0 \leq l_1 < l_2 \leq 2$ such that $F_2^{1l_1} = F_2^{1l_2}$. Then $f^{l_1} \equiv f^{l_2}$.

(ii) There are two distinct constants $\alpha, \beta \in \mathbb{C}\setminus\{0,1\}$ such that $F_2^{10} = \alpha F_2^{11} = \beta F_2^{12}$.

We may assume that $H_1 = \{\omega_0 = 0\}$, $H_2 = \{\omega_1 = 0\}$, $H_3 = \{\omega_0 - c\omega_1 = 0\}$ $(c \in \mathbb{C}\backslash\{0\})$. Then

$$\frac{f_0^0}{f_1^0} = \alpha\frac{f_0^1}{f_1^1} = \beta\frac{f_0^2}{f_1^2},$$

$$(f^1, H_3) = 0 \Leftrightarrow f_0^1 - cf_1^1 = 0 \Leftrightarrow (f_0^0 - c\alpha f_1^0)\left(\frac{f_1^1}{\alpha f_1^0}\right) = 0,$$

$$(f^2, H_3) = 0 \Leftrightarrow f_0^2 - cf_1^2 = 0 \Leftrightarrow (f_0^0 - c\beta f_1^0)\left(\frac{f_1^2}{\beta f_1^0}\right) = 0.$$

Hence $\{z \in \mathbb{C}^n : \nu_{(f^0,H_3),\leq k_{03}}(z) > 0\} \subset \bigcup_{i=0}^2 I(f^i)$. So that $N^{(1)}_{(f^0,H_3),\leq k_{03}}(r) = 0$, and $\nu_{(f^1,H_3)}(z) = \nu_{f_0^0 - c\alpha f_1^0}(z)$ and $\nu_{(f^2,H_3)}(z) = \nu_{f_0^0 - c\beta f_1^0}(z)$ for $z \notin I(f^0)\cup I(f^1)\cup I(f^2)$.

Thus, we have $\nu_{(f^1,H_3)}(z) = \nu_{f_0^0 - c\alpha f_1^0}(z)$ $(z \in \mathbb{C}^n)$ and $\nu_{(f^2,H_3)}(z) = \nu_{f_0^0 - c\beta f_1^0}(z)$ $(z \in \mathbb{C}^n)$.

Put $H'_3 = \{\omega_0 - c\alpha\omega_1 = 0\}$, $H''_3 = \{\omega_0 - c\beta\omega_1 = 0\}$. Then we have the following:

- H_3, H'_3, H''_3 are in general position.
- $\nu_{(f^0,H'_3)} = \nu_{(f^1,H_3)}$. This yields $\nu^{(1)}_{(f^0,H'_3),\le k_{13}} = \nu^{(1)}_{(f^1,H_3),\le k_{13}} = \nu^{(1)}_{(f^0,H_3),\le k_{03}}$
- $\nu_{(f^0,H''_3)} = \nu_{(f^2,H_3)}$. This yields $\nu^{(1)}_{(f^0,H''_3),\le k_{23}} = \nu^{(1)}_{(f^2,H_3),\le k_{23}} = \nu^{(1)}_{(f^0,H_3),\le k_{03}}$

By Lemma 3.9, we have

$$\begin{aligned} \Big\| \Big(3 - 1 - 1 - \sum_{j=0}^{2} \frac{1}{k_{j3}+1}\Big) T(r, f^0) \le (1 - \frac{1}{1+k_{03}}) N^{(1)}_{(f^0,H_3),\le k_{03}}(r) \\ + (1 - \frac{1}{1+k_{13}}) N^{(1)}_{(f^0,H'_3),\le k_{13}}(r) \\ + (1 - \frac{1}{1+k_{23}}) N^{(1)}_{(f^0,H''_3),\le k_{23}}(r) \\ + o(T(r, f^0)). \end{aligned}$$

$$\begin{aligned} \Rightarrow \Big(1 - \frac{3}{m+1}\Big) T(r, f^0) \le \Big(1 - \frac{1}{M+1}\Big) \Big(N^{(1)}_{(f^0,H_3),\le k_{03}}(r) \\ + N^{(1)}_{(f^0,H'_3),\le k_{13}}(r) + N^{(1)}_{(f^0,H''_3),\le k_{23}}(r) \Big) + o(T(r, f^0)). \end{aligned}$$

$$\begin{aligned} \Rightarrow \Big(1 - \frac{3}{m+1}\Big) T(r, f^0) \le \Big(1 - \frac{1}{M+1}\Big) \Big(N^{(1)}_{(f^0,H_3),\le k_{03}}(r) \\ + N^{(1)}_{(f^0,H_3),\le k_{03}}(r) + N^{(1)}_{(f^0,H_3),\le k_{03}}(r) \Big) + o(T(r, f^0)) \\ = 3(1 - \frac{1}{M+1}) N^{(1)}_{(f^0,H_3),\le k_{03}}(r) + o(T(r, f^0)). \end{aligned}$$

So we get

$$\Big(1 - \frac{3}{m+1}\Big) T(r, f^1) \le o(T(r, f^0)).$$

This is a contradiction. Case 2 of Theorem 2 is proved.

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