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ROBUST STABILIZATION OF LINEAR POLYTOPIC CONTROL SYSTEMS WITH MIXED DELAYS

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ABSTRACT. In this paper, a class of uncertain linear polytopic systems with mixed delays in state and control is studied. Using an improved parameter dependent Lyapunov-Krasovskii functional approach and linear matrix inequality technique, delay-dependent sufficient conditions for the robust stabilization of the system are first established in terms of the Mondié-Kharitonov type's linear matrix inequality (LMI) conditions. A numerical example is presented to demonstrate that the feedback control designed based on the obtained condition is effective, even though neither nominal control systems are controllable.

1. INTRODUCTION

The existence of time delays in both state and control may be the source of instability and serious deterioration in the performance of the closed-loop systems. The analysis of the stability and stabilization of time-delay control systems and the synthesis of feedback controllers for them are important both in theory and practice [3, 7, 12, 15]. When the system has only time delay in control input, one way to solve the stabilization problem of the system is the so-called reduction method [1], which reduces the system to a delay-free ordinary system by certain state transformation but the complete transformation can only be obtained for exactly known of the model and time delay. The results have been developed in [4, 9, 10, 11] for uncertain linear systems with state and input delays. However, stabilizing memory controllers obtained by this method are distributed and more complicated than memoryless feedback controllers, and therefore difficult to implement. Another type of uncertain delay systems, namely the linear system with polytopic-type uncertainties, has also received much attention in recent years [8, 14, 16]. However, the distributed delays are not taken into account in the mentioned papers. In practice, systems with distributed delays in both state and input have many important applications in various areas [6, 11, 13, 17]. Theoretically, systems with discrete and distributed delays in both state and control are

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much more complicated, especially for the case where the system matrices belong to some convex polytope. To the best of our knowledge, so far, no result on the stabilization for uncertain linear polytopic systems with discrete and distributed delays in both state and control is available in the literature, which is still open and remains unsolved. This motivates our present investigation.

In this paper, we develop robust stability problem for uncertain linear polytopic control systems with mixed delays. The novel feature of the results obtained in this paper is twofold. First, the system considered in this paper is convex polytopic uncertain subjected to discrete and distributed delays in both the state and control. Second, by employing an improved parameter-dependent Lyapunov Krasovskii functional and linear matrix inequality technique, delay-dependent sufficient conditions for the exponential stabilization of the system are first obtained in terms of the Mondié-Kharitonov type's LMI conditions [12]. The conditions do not require any assumption on the controllability of the nominal control system. The approach also allows to compute simultaneously the two bounds that characterize the exponential stability rate of the solution.

The paper is organized as follows: Section 2 presents notations, definitions and some well-known technical propositions needed for the proof of the main result. Delay-dependent robust stabilization conditions of the system are presented in Section 3. Numerical example is given in Section 4. The paper ends with conclusions and cited references.

2. Preliminaries

The following notations and definitions will be employed throughout this paper. R^+ denotes the set of all real non-negative numbers; R^n denotes the n-dimensional Euclidean space with the scalar product $\langle .,.\rangle$; $R^{n\times r}$ denotes the space of all matrices of $(n \times r)$-dimensions; A^T denotes the transpose of A, I_m denotes the identity matrix in $R^{m\times m}; \lambda(A)$ denotes the set of all eigenvalues of A, $\lambda_{\max}(A) = \max\{\mathrm{Re}\,\lambda : \lambda \in \lambda(A)\}$, $\lambda_{\min}(A) = \min\{\mathrm{Re}\,\lambda : \lambda \in \lambda(A)\}$; matrix $Q \geq 0$ ($Q > 0$, resp.) means Q is semi-positive definite matrix i.e. $\langle Qx, x\rangle \geq 0, \forall x \in R^n$ (positive definite, resp. i.e. $\langle Qx, x\rangle > 0, \forall x \in R^n, x \neq 0$), $A \geq B$ means $A - B \geq 0$; $C([a, b], R^n)$ denotes the set of all R^n-valued continuous functions on $[a, b]$; the segment of the trajectory $x(t)$ is denoted by $x_t = \{x(t + s) : s \in t \in [-h, 0]\}$ with its norm $\|x_t\| = \sup_{s\in[-h,0]} \|x(t + s)\|$.

Consider a linear uncertain polytopic system with discrete and distributed delays in state and control of the form

$$
\begin{aligned}
\dot{x}(t) &= A_0x(t) + A_1x(t-\tau) + A_2\int_{t-\tau}^{t} x(s)ds + B_0u(t) \\
&\qquad + B_1u(t-r) + B_2\int_{t-r}^{t} u(s)ds, t \in R^+, \qquad (2.1)\\
x(t) &= \phi(t), \quad t \in [-h, 0],
\end{aligned}
$$

where $h = \max\{\tau, r\}, x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control, $\tau \geq 0, r \geq 0$ are time delays; $\phi(t) \in C([-h, 0], R^n)$ is the initial function with the norm $\|\phi\| = \sup_{-h\leq s\leq 0} \|\phi(s)\|$. The matrices $A_k, B_k, k = 0, 1, 2$, are subject to uncertainties and belong to the polytope Ω given by

$$\Omega = \left\{ \left[A_k(\xi), B_k(\xi)\right] = \sum_{i=1}^{p} \xi_i [A_{ki}, B_{ki}], k = 0, 1, 2, \xi_i \geq 0, \sum_{i=1}^{p} \xi_i = 1 \right\},$$

where $p > 1$ and A_{kj}, B_{kj} are given real matrices.

In this paper, a memoryless parameter-dependent state feedback controller

$$u(t) = K(\xi)x(t), \tag{2.2}$$

is employed to stabilize (2.1).

Definition 2.1. Given $\alpha > 0$, system (2.1) is α-robustly stabilizable if there exists a linear state feedback control law (2.2) such that any solution of the closed-loop system satisfies the following inequality

$$\exists N > 0, \quad \|x(t, \phi)\| \leq N e^{-\alpha t} \|\phi\|, \quad t \geq 0.$$

The objective of this paper is to design the memoryless feedback control law that makes the system (2.1) robustly stabilizable. For this purpose, the following technical propositions are first introduced.

Proposition 2.1. (Schur complement lemma) [2]. *Let X, Y, Z be any matrices with appropriate dimensions, where $X = X^T$, $Y = Y^T > 0$. Then $X + Z^T Y^{-1} Z < 0$ if and only if*

$$\begin{pmatrix} X & Z^T \\ Z & -Y \end{pmatrix} < 0.$$

Proposition 2.2. (Matrix Cauchy inequality) *For any symmetric positive definite matrix $M \in R^{n\times n}$ and $x, y \in R^n$, we have*

$$2\langle x, y\rangle \leq \langle Mx, x\rangle + \langle M^{-1}y, y\rangle.$$

The above proposition is easily proved by completing the square.

Proposition 2.3. [5] *For any symmetric positive definite matrix $W \in R^{n\times n}$, scalar $\nu \geq 0$, and vector function $\omega : [0, \nu] \to R^n$ such that the integrals concerned are well defined, we have*

$$\left(\int_0^\nu \omega(s)ds\right)^T W \left(\int_0^\nu \omega(s)ds\right) \leq \nu \int_0^\nu \omega^T(s) W \omega(s)ds.$$

3. Main results

For given $\alpha > 0, \tau > 0, r > 0$, symmetric positive definite matrices $P_i, Q_i, R_i \in R^{n\times n}$, semi-positive definite matrix $S \in R^{n\times n}$ and matrices $Y_i \in R^{m\times n}, i =$

$1, 2, \dots, p$, we denote

$$P = \sum_{i=1}^{p} \xi_i P_i, \quad Q = \sum_{i=1}^{p} \xi_i Q_i, \quad R = \sum_{i=1}^{p} \xi_i R_i, \quad Y = \sum_{i=1}^{p} \xi_i Y_i,$$

$$G_{ij} = B_{0i} Y_j + Y_j^T B_{0i}^T + e^{2\alpha r}\left(B_{1i} B_{1j}^T + r B_{2i} B_{2j}^T\right),$$

$$\Gamma_{ij} = A_{0i} P_j + P_j A_{0i}^T + G_{ij} + Q_j + \tau R_j, \quad H_{ij} = \begin{pmatrix} A_{1i} P_j & A_{2i} P_j & Y_j^T \end{pmatrix},$$

$$D_j = diag\left\{ e^{-2\alpha\tau} Q_j, \quad \frac{1}{\tau} e^{-2\alpha\tau} R_j, \quad \mu I_m \right\}, \quad \mu = (1+r)^{-1},$$

$$\mathcal{M}_i(P_j, Q_j, R_j, Y_j) = \begin{pmatrix} \Gamma_{ij} & A_{1i} P_j & A_{2i} P_j & Y_j^T \\ P_j A_{1i}^T & -e^{-2\alpha\tau} Q_j & 0 & 0 \\ P_j A_{2i}^T & 0 & -\frac{1}{\tau} e^{-2\alpha\tau} R_j & 0 \\ Y_j & 0 & 0 & -\mu I_m \end{pmatrix} = \begin{pmatrix} \Gamma_{ij} & H_{ij} \\ H_{ij}^T & -D_j \end{pmatrix},$$

$$\mathbb{S} = \begin{pmatrix} S & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{N}(P_j) = \begin{pmatrix} P_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad i, j = 1, 2, \dots, p,$$

$$\lambda_-(P) = \min_{i=1,2,\dots,p} \{\lambda_{\min}(P_i)\}, \quad \lambda^+(P) = \max_{i=1,2,\dots,p} \{\lambda_{\max}(P_i)\},$$

$$\lambda^+(Q) = \max_{i=1,2,\dots,p} \{\lambda_{\max}(Q_i)\}, \quad \lambda^+(R) = \max_{i=1,2,\dots,p} \{\lambda_{\max}(R_i)\},$$

$$\lambda^+(Y^T Y) = \max_{i=1,2,\dots,p} \{\lambda_{\max}(Y_i^T Y_i)\}.$$

(3.1)

$$\alpha_1 = \frac{1}{\lambda^+(P)}, \quad \alpha_2 = \frac{1}{\lambda_-(P)} + \frac{\tau\lambda^+(Q) + \frac{1}{2}\tau^2\lambda^+(R) + \left(1 + \frac{1}{2}r^2\right)\lambda^+(Y^T Y)}{[\lambda_-(P)]^2}.$$

Theorem 3.1. *Given $\alpha > 0$, system (2.1) is α-robustly stabilizable if there exist symmetric positive definite matrices P_i, Q_i, R_i, matrices $Y_i, i = 1, 2, \dots, p$, and a semi-positive definite matrix S such that the following LMIs hold:*

i) $\mathcal{M}_i(P_i, Q_i, R_i, Y_i) + 2\alpha\mathcal{N}(P_i) < -\mathbb{S}, \quad i = 1, 2, \dots, p,$

ii) $\mathcal{M}_i(P_j, Q_j, R_j, Y_j) + \mathcal{M}_j(P_i, Q_i, R_i, Y_i) + 2\alpha\mathcal{N}(P_i + P_j) < \dfrac{2}{p-1}\mathbb{S},$
$i = 1, \dots, p-1, j = i+1, \dots, p.$

The feedback stabilizing control is given by

$$u(t) = Y P^{-1} x(t), \quad t \geq 0.$$

Moreover, every solution $x(t, \phi)$ satisfies

$$\|x(t, \phi)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}} e^{-\alpha t} \|\phi\|, \quad t \geq 0.$$

Proof. Because $P_i > 0, \xi_i \geq 0, i = 1, 2, \dots, p$ and $\sum_{i=1}^{p} \xi_i = 1$ we have $P = \sum_{i=1}^{p} \xi_i P_i$ is symmetric positive definite. Denote $X = P^{-1}, \bar{Q} = XQX, \bar{R} =$

$XRX, K = YX$, and consider the following parameter-dependent Lyapunov-Krasovskii functional for the closed-loop system of (2.1):

$$V(x_t) = V_1 + V_2 + V_3 + V_4 + V_5, \tag{3.2}$$

where

$$\begin{aligned}
V_1 &= x^T(t)Xx(t)\\
V_2 &= \int_{-\tau}^{0} e^{2\alpha s} x^T(t+s)\bar{Q}x(t+s)ds,\\
V_3 &= \int_{-\tau}^{0}\int_{s}^{0} e^{2\alpha\zeta} x^T(t+\zeta)\bar{R}x(t+\zeta)d\zeta ds,\\
V_4 &= \int_{-r}^{0} e^{2\alpha s} x^T(t+s)K^TKx(t+s)ds,\\
V_5 &= \int_{-r}^{0}\int_{s}^{0} e^{2\alpha\zeta} x^T(t+\zeta)K^TKx(t+\zeta)d\zeta ds.
\end{aligned}$$

It can be verified from (3.2) that

$$\alpha_1\|x(t)\|^2 \le V(x_t) \le \alpha_2\|x_t\|^2, \quad t \ge 0, \tag{3.3}$$

where α_1, α_2 are defined in (3.1).

Taking derivative of V_1 along solutions of the closed-loop system, we get

$$\begin{aligned}
\dot{V}_1 = x^T(t)\Big[A_0^TX + XA_0 + X(B_0Y + Y^TB_0^T)X\Big]x(t)&\\
+ 2x^T(t)XA_1x(t-\tau) + 2x^T(t)XB_1Kx(t-r)&\\
+ 2x^T(t)XA_2\int_{t-\tau}^{t} x(s)ds + 2x^T(t)XB_2\int_{t-r}^{t} Kx(s)ds.&
\end{aligned} \tag{3.4}$$

Applying Proposition 2.2 and 2.3 we have

$$\begin{aligned}
2x^T(t)XA_1x(t-\tau) &\le e^{2\alpha\tau}x^T(t)XA_1\bar{Q}^{-1}A_1^TXx(t) + e^{-2\alpha\tau}x^T(t-\tau)\bar{Q}x(t-\tau),\\
2x^T(t)XB_1u(t-r) &\le e^{2\alpha r}x^T(t)XB_1B_1^TXx(t) + e^{-2\alpha r}\|Kx(t-r)\|^2.
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
2x^T(t)XA_2\int_{t-\tau}^{t} x(s)ds &\le \tau e^{2\alpha\tau}x^T(t)XA_2\bar{R}^{-1}A_2^TXx(t)\\
&\quad + \frac{1}{\tau}e^{-2\alpha\tau}\left(\int_{t-\tau}^{t} x(s)ds\right)^T \bar{R}\left(\int_{t-\tau}^{t} x(s)ds\right)\\
&\le \tau e^{2\alpha\tau}x^T(t)XA_2\bar{R}^{-1}A_2^TXx(t)\\
&\quad + e^{-2\alpha\tau}\int_{t-\tau}^{t} x^T(s)\bar{R}x(s)ds.
\end{aligned} \tag{3.6}$$

$$
\begin{aligned}
2x^T(t)XB_2\int_{t-r}^t Kx(s)ds &\le re^{2\alpha r}x^T(t)XB_2B_2^TXx(t)\\
&\quad + \frac{1}{r}e^{-2\alpha r}\left(\int_{t-r}^t Kx(s)ds\right)^T\left(\int_{t-r}^t Kx(s)ds\right)\\
&\le re^{2\alpha r}x^T(t)XB_2B_2^TXx(t)\\
&\quad + e^{-2\alpha r}\int_{t-r}^t \|Kx(s)\|^2 ds.
\end{aligned} \tag{3.7}
$$

Therefore, from (3.4) to (3.7) we have

$$
\begin{aligned}
\dot{V}_1 \le\ & x^T(t)\Big[A_0^TX + XA_0 + X(B_0Y + Y^TB_0^T)X\Big]x(t)\\
&+ e^{2\alpha\tau}x^T(t)XA_1\bar{Q}^{-1}A_1^TXx(t) + e^{-2\alpha\tau}x^T(t-\tau)\bar{Q}x(t-\tau)\\
&+ e^{2\alpha r}x^T(t)XB_1B_1^TXx(t) + e^{-2\alpha r}\|Kx(t-r)\|^2\\
&+ \tau e^{2\alpha\tau}x^T(t)XA_2\bar{R}^{-1}A_2^TXx(t) + e^{-2\alpha\tau}\int_{t-\tau}^t x^T(s)\bar{R}x(s)ds\\
&+ re^{2\alpha r}x^T(t)XB_2B_2^TXx(t) + e^{-2\alpha r}\int_{t-r}^t \|Kx(s)\|^2 ds.
\end{aligned} \tag{3.8}
$$

Next, taking derivative of $V_i, i = 2,3,4,5$, along solutions of the closed-loop system respectively, we obtain

$$
\begin{aligned}
\dot{V}_2 &= x^T(t)\bar{Q}x(t) - e^{-2\alpha\tau}x^T(t-\tau)\bar{Q}x(t-\tau) - 2\alpha V_2,\\
\dot{V}_3 &= \tau x^T(t)\bar{R}x(t) - \int_{-\tau}^0 e^{2\alpha s}x^T(t+s)\bar{R}x(t+s)ds - 2\alpha V_3\\
&\le \tau x^T(t)\bar{R}x(t) - e^{-2\alpha\tau}\int_{t-\tau}^t x^T(s)\bar{R}x(s)ds - 2\alpha V_3.
\end{aligned} \tag{3.9}
$$

$$
\begin{aligned}
\dot{V}_4 &= \|Kx(t)\|^2 - e^{-2\alpha r}\|Kx(t-r)\|^2 - 2\alpha V_4\\
&= x^T(t)K^TKx(t) - e^{-2\alpha r}\|Kx(t-r)\|^2 - 2\alpha V_4,\\
\dot{V}_5 &= r\|Kx(t)\|^2 - \int_{-r}^0 e^{2\alpha s}\|Kx(t+s)\|^2 - 2\alpha V_5\\
&\le rx^T(t)K^TKx(t) - e^{-2\alpha r}\int_{t-r}^t \|Kx(s)\|^2 ds - 2\alpha V_5.
\end{aligned} \tag{3.10}
$$

Combining (3.8) - (3.10) gives

(3.11)

$$
\begin{aligned}
\dot{V}(x_t) + 2\alpha V(x_t) \leq\ & x^T(t)\left(A_0^T X + X A_0 + 2\alpha X\right)x(t) \\
& + x^T(t)X\left[B_0 Y + Y^T B_0^T + (1+r)Y^T Y + Q + \tau R\right]Xx(t) \\
& + e^{2\alpha\tau} x^T(t)X\left(A_1\bar{Q}^{-1}A_1^T + \tau A_2 \bar{R}^{-1} A_2^T\right)Xx(t) \\
& + e^{2\alpha r} x^T(t)X\left(B_1 B_1^T + r B_2 B_2^T\right)Xx(t) \\
=\ & \eta^T(t)\left(A_0 P + P A_0^T + 2\alpha P + G + Q + \tau R + H D^{-1} H^T\right)\eta(t), \\
=\ & \eta^T(t)\left(\Gamma + 2\alpha P + H D^{-1} H^T\right)\eta(t),
\end{aligned}
$$

where

$$
\begin{aligned}
\eta(t) &= Xx(t), \\
G &= B_0 Y + Y^T B_0^T + e^{2\alpha r}\left(B_1 B_1^T + r B_2 B_2^T\right), \\
\Gamma &= A_0 P + P A_0^T + G + Q + \tau R, \\
H &= \begin{pmatrix} A_1 P & A_2 P & Y^T \end{pmatrix}, \\
D &= diag\left\{ e^{-2\alpha\tau} Q, \quad \frac{1}{\tau} e^{-2\alpha\tau} R, \quad \mu I_m \right\}.
\end{aligned}
$$

On the other hand, since $[A_k, B_k], k = 0, 1, 2$, belong to Ω, $P = \sum_{i=1}^p \xi_i P_i$, $Q = \sum_{i=1}^p \xi_i Q_i$, $R = \sum_{i=1}^p \xi_i R_i$, and $\sum_{i=1}^p \xi_i = 1$, we have

$$
\begin{aligned}
\begin{pmatrix} \Gamma & H \\ H^T & -D \end{pmatrix} + 2\alpha \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} =\ & \sum_{i=1}^p \xi_i^2 \left\{ \begin{pmatrix} \Gamma_{ii} & H_{ii} \\ H_{ii}^T & -D_i \end{pmatrix} + 2\alpha \begin{pmatrix} P_i & 0 \\ 0 & 0 \end{pmatrix} \right\} \\
& + \sum_{i=1}^{p-1} \sum_{j=i+1}^{p} \xi_i \xi_j \Bigg[\begin{pmatrix} \Gamma_{ij} + \Gamma_{ji} & H_{ij} + H_{ji} \\ H_{ij}^T + H_{ji}^T & -(D_i + D_j) \end{pmatrix} \\
& + 2\alpha \begin{pmatrix} P_i + P_j & 0 \\ 0 & 0 \end{pmatrix} \Bigg] \\
=\ & \sum_{i=1}^p \xi_i^2 \Big[\mathcal{M}_i(P_i, Q_i, R_i, Y_i) + 2\alpha \mathcal{N}(P_i) \Big] \\
& + \sum_{i=1}^{p-1} \sum_{j=i+1}^{p} \xi_i \xi_j \Big[\mathcal{M}_i(P_j, Q_j, R_j, Y_j) \\
& + \mathcal{M}_j(P_i, Q_i, R_i, Y_i) + 2\alpha \mathcal{N}(P_i + P_j) \Big].
\end{aligned}
$$

By the conditions (i) and (ii) of the theorem, we obtain

$$
\begin{pmatrix} \Gamma & H \\ H^T & -D \end{pmatrix} + 2\alpha \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} < -\sum_{i=1}^p \xi_i^2 \mathbb{S} + \frac{2}{p-1} \sum_{i=1}^{p-1} \sum_{j=i+1}^{p} \xi_i \xi_j \mathbb{S}
$$

$$= -\frac{1}{p-1}\Big[(p-1)\sum_{i=1}^{p}\xi_i^2 - 2\sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\xi_i\xi_j\Big]\mathbb{S}.$$

Since

$$(p-1)\sum_{i=1}^{p}\xi_i^2 - 2\sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\xi_i\xi_j = \sum_{i=1}^{p-1}\sum_{j=i+1}^{p}(\xi_i-\xi_j)^2 \geq 0,$$

we have

$$\begin{pmatrix}\Gamma + 2\alpha P & H\\ H^T & -D\end{pmatrix} < 0. \tag{3.12}$$

By Proposition 2.1, inequality (3.12) implies that

$$\Gamma + 2\alpha P + HD^{-1}H^T < 0.$$

Therefore, the condition (3.11) becomes

$$\dot{V}(x_t) + 2\alpha V(x_t) \leq 0, \quad \forall t \geq 0,$$

and hence

$$V(x_t) \leq V(\phi)e^{-2\alpha t} \leq \alpha_2\|\phi\|^2 e^{-2\alpha t}, \quad t \geq 0.$$

Taking (3.3) into account, we finally obtain

$$\|x(t,\phi)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}}e^{-\alpha t}\|\phi\|, \quad t \geq 0,$$

where α_1, α_2 are defined in (3.1). The proof of the theorem is complete.

Remark 3.1. It is worth noting that the condition (i) means the asymptotic stability of each $i^{th}-$subsystem, the condition (ii) implies the asymptotic stability of the $ij^{th}-$subsystems and if $p = 1$ this condition is automatically removed. Thus, Theorem 3.1 includes the result of [4, 10, 12] for the systems without polytope type uncertainties ($p = 1$) and of [8, 16] for polytopic systems without distributed delays ($A_2 = B_2 = 0$). Moreover, the memoryless feedback stabilizing control for the system (2.1) can be designed even when neither the nominal control system (A_{0i}, B_{0i}) nor $(A_{0i} + A_{1i}, B_{0i})$ are controllable.

4. Numerical example

Consider the control system (2.1), where $p = 3, \tau = 1, r = 1$ and

$$A_{01} = \begin{pmatrix}-10 & 1\\ 0 & -10\end{pmatrix}, \quad A_{02} = \begin{pmatrix}-9 & 2\\ 0 & -15\end{pmatrix}, \quad A_{03} = \begin{pmatrix}-8 & 1\\ 0 & -12\end{pmatrix},$$

$$A_{11} = \begin{pmatrix}-2 & 1\\ 0 & 1\end{pmatrix}, \quad A_{12} = \begin{pmatrix}1 & -1\\ 0 & -4\end{pmatrix}, \quad A_{13} = \begin{pmatrix}2 & 0\\ 0 & -1\end{pmatrix},$$

$$A_{21} = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, \quad A_{22} = \begin{pmatrix}1 & 1\\ 0 & -4\end{pmatrix}, \quad A_{23} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

$$B_{01} = \begin{pmatrix}1\\ 0\end{pmatrix}, \quad B_{02} = \begin{pmatrix}2\\ 0\end{pmatrix}, \quad B_{03} = \begin{pmatrix}3\\ 0\end{pmatrix}, \quad B_{11} = \begin{pmatrix}0\\ 1\end{pmatrix},$$

$$B_{12} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \quad B_{13} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad B_{21} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad B_{22} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad B_{23} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

By Kalman rank condition, neither linear control system (A_{0i}, B_{0i}) nor $(A_{0i} + A_{1i}, B_{0i})$ are controllable. However, for $\alpha = 0.5$, the conditions (i) and (ii) in Theorem 3.1 are satisfied. By using LMI toolbox of Matlab, we find that all LMIs in Theorem 3.1 are feasible with

$$\mathbb{S} = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 13.4444 & -0.1241 \\ -0.1241 & 8.4191 \end{pmatrix},$$
$$P_2 = \begin{pmatrix} 4.7545 & 1.3646 \\ 1.3646 & 14.0144 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 9.0987 & -0.0313 \\ -0.0313 & 2.6604 \end{pmatrix},$$
$$Q_1 = \begin{pmatrix} 67.5881 & -10.8587 \\ -10.8587 & 34.2100 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 8.7531 & 6.0254 \\ 6.0254 & 81.4335 \end{pmatrix},$$
$$Q_3 = \begin{pmatrix} 16.3252 & -2.2418 \\ -2.2418 & 5.2477 \end{pmatrix}, \quad R_1 = \begin{pmatrix} 63.8807 & -5.0032 \\ -5.0032 & 32.7827 \end{pmatrix},$$
$$R_2 = \begin{pmatrix} 8.9332 & 5.7459 \\ 5.7459 & 75.9097 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 22.4391 & 0.0307 \\ 0.0307 & 3.3582 \end{pmatrix},$$
$$Y_1 = \begin{pmatrix} -0.3487 & -0.1106 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} -1.1660 & 0.1127 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} -0.4223 & -0.1062 \end{pmatrix}.$$

The state feedback stabilizing control is given by

$$\begin{aligned} u(t) &= YP^{-1}x(t) \\ &= (\xi_1 Y_1 + \xi_2 Y_2 + \xi_3 Y_3) \times (\xi_1 P_1 + \xi_2 P_2 + \xi_3 P_3)^{-1} x(t) \\ &= \frac{1}{p_1 p_3 - p_2^2} \begin{pmatrix} z_1 p_3 - z_2 p_2 & z_2 p_1 - z_1 p_2 \end{pmatrix} x(t), \end{aligned}$$

where

$$\begin{aligned} z_1 &= -0.3847\xi_1 - 1.1660\xi_2 - 0.4223\xi_3, \\ z_2 &= -0.1106\xi_1 + 0.1127\xi_2 - 0.1062\xi_3, \\ p_1 &= 13.4444\xi_1 + 4.7545\xi_2 + 9.0987\xi_3, \\ p_2 &= -0.1241\xi_1 + 1.3464\xi_2 - 0.0313\xi_3, \\ p_3 &= 8.4191\xi_1 + 14.0144\xi_2 + 2.6604\xi_3. \end{aligned}$$

Moreover, we find $N = \sqrt{\frac{\alpha_2}{\alpha_1}} = 15.8316$, and every solution $x(t, \phi)$ of the closed-loop system satisfies

$$\|x(t, \phi)\| \leq 15.8316 e^{-0.5t} \|\phi\|, \quad t \geq 0.$$

5. Conclusion

This paper has proposed new sufficient conditions for the robust stabilization of uncertain linear polytopic systems with discrete and distributed delays in both state and control. Based on an improved Lyapunov-Krasovskii parameter-dependent functional, delay-dependent exponential stabilization conditions of the system are derived in terms of Mondié-Kharitonov type's LMI, which allows to compute simultaneously the two bounds that characterize the exponential stability of the solution. A numerical example to illustrate the obtained result is given.

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