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CONVERGENCE THEOREMS OF MULTI-STEP ITERATION WITH ERRORS FOR FINITE FAMILIES OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. The aim of this paper is to study weak and strong convergence of common fixed points for multi-step iterative scheme with errors for finite families of asymptotically nonexpansive mappings in real uniformly convex Banach spaces. Our results extend and improve the corresponding results of Khan and Takahashi [13], Schu [22], Takahashi and Tamura [26], Rhoades [21], Xu and Noor [28], Khan and Fukhar-ud-din [12], Plubtieng et al. [20] and many others.

1. Introduction

Let K be a nonempty subset of a real Banach space E. A self-mapping $T\colon K \to K$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in K$. T is said to be asymptotically nonexpansive [8] if there exists a sequence $\{r_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} r_n = 0$ such that

$$\|T^n x - T^n y\| \leq (1 + r_n)\|x - y\|$$

for all $x, y \in K$ and $n \geq 1$.

The class of asymptotically nonexpansive mappings which is an important generalization of that of nonexpansive mappings was introduced by Goebel and Kirk [8]. Iteration processes for nonexpansive and asymptotically nonexpansive mappings in Banach spaces including Mann [15] and Ishikawa [11] iteration process have been studied extensively by many authors to solve operators as well as variational inequalities, see [2-28] and references therein.

In 2000, Noor [16] introduced a three step iterative scheme and studied the approximate solution of variational inclusion in Hilbert spaces by using the techniques of updating the solution and auxiliary principle. Glowinski and Le Tallec [7] used three step iterative schemes to find the approximate solution of the elastoviscoplasticity problem, liquid crystal theory, and eigenvalue computation.

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It has been shown [7] that three step iterative scheme gives better numerical results than the two step and one step approximate iterations. Thus we conclude that three step scheme plays an important and significant role in solving various problems, which arise in pure and applied sciences. Recently, Xu and Noor [28] studied a three step scheme to approximate fixed points of asymptotically nonexpansive mappings in Banach spaces. In 2004, Cho et al. [5] extended the work of Xu and Noor [28] to the three step iterative scheme with errors in Banach spaces and gave weak and strong convergence theorems for asymptotically nonexpansive mappings in a Banach space. Moreover, Suantai [25] gave weak and strong convergence theorems for a new three step iterative scheme of asymptotically nonexpansive mappings. More recently, Plubtieng et al. [20] introduced three step iterative scheme with errors for three asymptotically nonexpansive mappings and established strong convergence of this scheme to common fixed point of three asymptotically nonexpansive mappings.

Inspired and motivated by the above facts, a new multi-step iteration scheme with errors for finite family of asymptotically nonexpansive mappings is introduced and strong and weak convergence theorems for this scheme to common fixed point are proved.

Let K be a nonempty closed subset of a normed space E and let $T_1, T_2 \dots, T_N \colon K \to K$ be N asymptotically nonexpansive mappings. For a given $x_1 \in K$ and a fixed $N \in \mathbb{N}$ ($\mathbb{N}$ denote the set of all positive integers), compute the sequence $\{x_n\}$ by

$$\begin{aligned} x_n^1 &= \alpha_n^1 T_1^n x_n + \beta_n^1 x_n + \gamma_n^1 u_n^1, \\ x_n^2 &= \alpha_n^2 T_2^n x_n^1 + \beta_n^2 x_n + \gamma_n^2 u_n^2, \\ \dots &= \dots \\ \dots &= \dots \\ x_{n+1} = x_n^N &= \alpha_n^N T_N^n x_n^{N-1} + \beta_n^N x_n + \gamma_n^N u_n^N, \end{aligned} \tag{1.1}$$

where $\{\alpha_n^i\}$, $\{\beta_n^i\}$, $\{\gamma_n^i\}$ are appropriate sequences in $[0,1]$ with $\alpha_n^i + \beta_n^i + \gamma_n^i = 1$ for each $i \in \{1, 2, \dots, N\}$ and $\{u_n^1\}, \{u_n^2\}, \dots, \{u_n^N\}$ are bounded sequences in K.

In this paper, we extend the scheme of Xu and Noor [28] to N-step iteration scheme with errors for finite families of asymptotically nonexpansive mappings and without boundedness condition on K. Our results presented in this paper extend and improve the corresponding results in [12, 13, 20, 21, 22, 26, 28].

2. Preliminaries

For the sake of convenience, we gather some basic definitions and set out our terminology needed in the sequel.

Definition 2.1. (see [8]) A Banach space X is said to be uniformly convex if the modulus of convexity of X

$$\delta_X(\varepsilon) = \inf\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x-y\| = \varepsilon\} > 0$$

for all $0 < \varepsilon \leq 2$ (i.e. $\delta_X(\varepsilon)$ is a function $(0,2] \to (0,1]$).

Definition 2.2. (see [17]) A Banach space X is said to satisfy Opial's condition [17] if for any sequence $\{x_n\}$ in X, $x_n \to x$ weakly implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in X$ with $y \neq x$.

Definition 2.3. (i) A mapping $T\colon K \to K$ with $F(T) \neq \emptyset$ is said to satisfy condition (A) [24] on K if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that for all $x \in K$, $\|x - Tx\| \geq f(d(x, F(T)))$ where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$, where $F(T)$ denotes the set of fixed points of T.

(ii) A family $\{T_1, T_2, \ldots, T_N\}$ of N self mappings on K with $\mathcal{F} = \cap_{i=1}^N F(T_i) \neq \emptyset$ is said to satisfy condition (B) on K if there exist f and d as in (i) such that $\max_{1\leq i\leq N}\{\|x - T_i x\|\} \geq f(d(x, \mathcal{F}))$ for all $x \in K$.

Note that condition (B) reduces to condition (A), when $T_i = T$ for all $i = 1, 2, \ldots, N$.

Lemma 2.4. *Let $\{a_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$ and $\{r_n\}_{n=1}^\infty$ be sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 + r_n)a_n + \beta_n, \quad \forall n \in N.$$

If $\sum_{n=1}^\infty r_n < \infty$ and $\sum_{n=1}^\infty \beta_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.5. (see [29]) *Let $p > 1$ and $R > 1$ be two fixed numbers and E a Banach space. Then E is uniformly convex if and only if there exists a continuous, strictly increasing and convex function $g\colon [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that $\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda)g(\|x - y\|)$ for all $x, y \in B_R(0) = \{x \in E : \|x\| \leq R\}$ and $\lambda \in [0,1]$, where $W_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$.*

Lemma 2.6. (see [9]) *Let E be a uniformly convex Banach space satisfying Opial's condition and C be a nonempty closed and convex subset of E. Let $T\colon C \to C$ be an asymptotically nonexpansive mapping. Then $I - T$ is demiclosed with respect to zero, i.e. for any sequence $\{x_n\}$ in C with $x_n \to x$ weakly and $x_n - Tx_n \to 0$, we have $x = Tx$.*

Lemma 2.7. *Let E be a normed linear space and K be a nonempty closed and convex subset of E. Let $T_1, T_2, \ldots, T_N\colon K \to K$ be N uniformly L-Lipschitzian mappings. Let $\{x_n\}$ be the sequence defined by (1.1) with sequences $\{u_n^i\}$ in K for all $i = 1, 2, \ldots, N$ and $\{\alpha_n^i\}$, $\{\beta_n^i\}$ and $\{\gamma_n^i\}$ are sequences in $[0,1]$ satisfying $\alpha_n^i + \beta_n^i + \gamma_n^i = 1$ for all $i = 1, 2, \ldots, N$. Set $c_n^i = \|x_n - T_i^n x_n\|$ for all $i = 1, 2, \ldots, N$. If $\lim_{n\to\infty} \|x_n - T_i^n x_n\| = 0$ and $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, then $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$.*

Proof. Since T_i is uniformly L-Lipschitzian for all $i = 1, 2, \ldots, N$, we have

$$\|x_{n+1} - T_i x_{n+1}\| \leq \|x_{n+1} - T_i^{n+1} x_{n+1}\| + \|T_i^{n+1} x_{n+1} - T_i x_{n+1}\|$$

$$
\begin{aligned}
&\leq c_{n+1}^i + L\|T_i^n x_{n+1} - x_{n+1}\| \\
&\leq c_{n+1}^i + L\{\|x_{n+1} - x_n\| + \|x_n - T_i^n x_n\| + \|T_i^n x_n - T_i^n x_{n+1}\|\} \\
&\leq c_{n+1}^i + L\{\|x_{n+1} - x_n\| + c_n^i + L\|x_{n+1} - x_n\|\} \\
&\leq c_{n+1}^i + L(L+1)\|x_{n+1} - x_n\| + Lc_n^i \to 0 \ \ as \ \ n \to \infty.
\end{aligned}
$$

This completes the proof. □

Remark 2.8. Lemma 2.7 generalizes the corresponding Lemma of Schu [23] for one mapping. Further, if $\mathcal{F} = \cap_{i=1}^N F(T_i) \neq \emptyset$ and $\lim_{n\to\infty} \|x_n - T_i^n x_n\| = 0$, then we have $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

3. Main results

Theorem 3.1. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $T_1, T_2, \dots, T_N \colon K \to K$ be N asymptotically nonexpansive mappings and $\mathcal{F} = \cap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{\alpha_n^i\}$, $\{\beta_n^i\}$ and $\{\gamma_n^i\}$ are sequences in $[0,1]$ with $\alpha_n^i + \beta_n^i + \gamma_n^i = 1$ for all $i = 1, 2, \dots, N$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by* (1.1). *Then*

(i) $\|x_{n+1} - x^*\| = \|x_n^N - x^*\| \leq (1 + b_n^{N-1})\|x_n - x^*\| + d_n^{N-1}$, *for all $n \geq 1$ and $x^* \in \mathcal{F}$, and for some sequences $\{b_n^i\}$ and $\{d_n^i\}$ for all $i = 1, 2, \dots, N$ of numbers such that $\sum_{n=1}^\infty b_n^i < \infty$ and $\sum_{n=1}^\infty d_n^i < \infty$.*

(ii) *There exists a constant $M > 0$ such that $\|x_{n+m} - x^*\| \leq M.\|x_n - x^*\| + M.\sum_{k=n}^{n+m-1} d_k^{N-1}$ for all $n, m \geq 1$ and $x^* \in \mathcal{F}$.*

Proof. (i) Let $x^* \in \mathcal{F}$, then from (1.1) we have

$$
\begin{aligned}
\|x_n^1 - x^*\| &= \|\alpha_n^1 T_1^n x_n + \beta_n^1 x_n + \gamma_n^1 u_n^1 - x^*\| \\
&\leq \alpha_n^1 \|T_1^n x_n - x^*\| + \beta_n^1 \|x_n - x^*\| + \gamma_n^1 \|u_n^1 - x^*\| \\
&\leq \alpha_n^1 (1 + r_n^1)\|x_n - x^*\| + \beta_n^1 \|x_n - x^*\| + \gamma_n^1 \|u_n^1 - x^*\| \\
&\leq (1 - \beta_n^1)(1 + r_n^1)\|x_n - x^*\| + \beta_n^1 (1 + r_n^1)\|x_n - x^*\| + \gamma_n^1 \|u_n^1 - x^*\| \\
&\leq (1 + r_n^1)\|x_n - x^*\| + \gamma_n^1 \|u_n^1 - x^*\| \\
&\leq (1 + r_n^1)\|x_n - x^*\| + d_n^0
\end{aligned}
$$

where $d_n^0 = \gamma_n^1 \|u_n^1 - x^*\|$. Since $\sum_{n=1}^\infty \gamma_n^1 < \infty$, then $\sum_{n=1}^\infty d_n^0 < \infty$. Next, we note that

$$
\begin{aligned}
\|x_n^2 - x^*\| &= \|\alpha_n^2 T_2^n x_n^1 + \beta_n^2 x_n + \gamma_n^2 u_n^2 - x^*\| \\
&\leq \alpha_n^2 \|T_2^n x_n^1 - x^*\| + \beta_n^2 \|x_n - x^*\| + \gamma_n^2 \|u_n^2 - x^*\| \\
&\leq \alpha_n^2 (1 + r_n^2)\|x_n^1 - x^*\| + \beta_n^2 \|x_n - x^*\| + \gamma_n^2 \|u_n^2 - x^*\| \\
&\leq \alpha_n^2 (1 + r_n^2)[(1 + r_n^1)\|x_n - x^*\| + d_n^0] + \beta_n^2 \|x_n - x^*\| + \gamma_n^2 \|u_n^2 - x^*\| \\
&\leq [(1 + r_n^1)(1 + r_n^2)\alpha_n^2 + \beta_n^2]\|x_n - x^*\| + \alpha_n^2 (1 + r_n^2) d_n^0 + \gamma_n^2 \|u_n^2 - x^*\| \\
&\leq (\alpha_n^2 + \beta_n^2)(1 + r_n^1)(1 + r_n^2)\|x_n - x^*\| + \alpha_n^2 (1 + r_n^2) d_n^0 + \gamma_n^2 \|u_n^2 - x^*\|
\end{aligned}
$$

$$\begin{aligned}&\leq (1+r_n^1+r_n^2+r_n^1r_n^2)\|x_n-x^*\|+\alpha_n^2(1+r_n^2)d_n^0+\gamma_n^2\|u_n^2-x^*\|\\&\leq (1+b_n^1)\|x_n-x^*\|+d_n^1,\end{aligned}$$

where $d_n^1=\alpha_n^2(1+r_n^2)d_n^0+\gamma_n^2\|u_n^2-x^*\|$ and $b_n^1=(1+r_n^1+r_n^2+r_n^1r_n^2)$. Since $\sum_{n=1}^\infty d_n^0<\infty$, $\sum_{n=1}^\infty \gamma_n^2<\infty$, $\sum_{n=1}^\infty r_n^i<\infty$ for $i=1,2$ and so $\sum_{n=1}^\infty d_n^1<\infty$ and $\sum_{n=1}^\infty b_n^1<\infty$. Similarly, we have

$$\begin{aligned}\|x_n^3-x^*\|&=\|\alpha_n^3T_3^nx_n^2+\beta_n^3x_n+\gamma_n^3u_n^3-x^*\|\\&\leq \alpha_n^3\|T_3^nx_n^2-x^*\|+\beta_n^3\|x_n-x^*\|+\gamma_n^3\|u_n^3-x^*\|\\&\leq \alpha_n^3(1+r_n^3)\|x_n^2-x^*\|+\beta_n^3\|x_n-x^*\|+\gamma_n^3\|u_n^3-x^*\|\\&\leq \alpha_n^3(1+r_n^3)[(1+b_n^1)\|x_n-x^*\|+d_n^1]+\beta_n^3\|x_n-x^*\|+\gamma_n^3\|u_n^3-x^*\|\\&\leq [\alpha_n^3(1+r_n^3)(1+b_n^1)+\beta_n^3]\|x_n-x^*\|+\alpha_n^3(1+r_n^3)d_n^1+\gamma_n^3\|u_n^3-x^*\|\\&\leq (\alpha_n^3+\beta_n^3)(1+b_n^1)(1+r_n^3)\|x_n-x^*\|+\alpha_n^3(1+r_n^3)d_n^1+\gamma_n^3\|u_n^3-x^*\|\\&\leq (1+b_n^1)(1+r_n^3)\|x_n-x^*\|+d_n^2\\&\leq (1+b_n^2)\|x_n-x^*\|+d_n^2,\end{aligned}$$

where $b_n^2=b_n^1+r_n^3+b_n^1r_n^3$ and $d_n^2=\alpha_n^3(1+r_n^3)d_n^1+\gamma_n^3\|u_n^3-x^*\|$. Since $\sum_{n=1}^\infty b_n^1<\infty$, $\sum_{n=1}^\infty r_n^3<\infty$, $\sum_{n=1}^\infty d_n^1<\infty$ and $\sum_{n=1}^\infty \gamma_n^3<\infty$, so $\sum_{n=1}^\infty b_n^2<\infty$ and $\sum_{n=1}^\infty d_n^2<\infty$.

By continuing the above process, there exist nondecreasing sequences $\{d_n^{l-1}\}$ and $\{b_n^{l-1}\}$ such that $\sum_{n=1}^\infty d_n^{l-1}<\infty$ and $\sum_{n=1}^\infty b_n^{l-1}<\infty$ and

$$\|x_n^i-x^*\|\leq (1+b_n^{i-1})\|x_n-x^*\|+d_n^{i-1},\quad \forall n\geq 1,\quad \forall i=1,2,\ldots,N.$$

Thus

$$\|x_{n+1}-x^*\|=\|x_n^N-x^*\|\leq (1+b_n^{N-1})\|x_n-x^*\|+d_n^{N-1},\quad \forall n\in N.$$

This completes the proof of (i).

(ii) Since $1+x\leq e^x$ for all $x>0$, therefore from (i) it can be obtained that

$$\begin{aligned}\|x_{n+m}-x^*\|&\leq (1+b_{n+m-1}^{N-1})\|x_{n+m-1}-x^*\|+d_{n+m-1}^{N-1}\\&\leq e^{b_{n+m-1}^{N-1}}\|x_{n+m-1}-x^*\|+d_{n+m-1}^{N-1}\\&\leq e^{(b_{n+m-1}^{N-1}+b_{n+m-2}^{N-1})}\|x_{n+m-2}-x^*\|+e^{b_{n+m-1}^{N-1}}d_{n+m-2}^{N-1}+d_{n+m-1}^{N-1}\\&\leq e^{(b_{n+m-1}^{N-1}+b_{n+m-2}^{N-1})}\|x_{n+m-2}-x^*\|+e^{b_{n+m-1}^{N-1}}(d_{n+m-1}^{N-1}+d_{n+m-2}^{N-1})\\&\leq \ldots\ldots\\&\leq \ldots\ldots\\&\leq e^{\sum_{k=n}^{n+m-1}b_k^{N-1}}\|x_n-x^*\|+e^{\sum_{k=n}^{n+m-1}b_k^{N-1}}.\sum_{k=n}^{n+m-1}d_k^{N-1}\end{aligned}$$

$$\leq M.\|x_n - x^*\| + M.\sum_{k=n}^{n+m-1} d_k^{N-1}, \text{ where } M = e^{\sum_{k=n}^{\infty} b_k^{N-1}}.$$

This completes the proof of (ii). □

Theorem 3.2. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $T_1, T_2, \dots, T_N\colon K \to K$ be N uniformly continuous asymptotically nonexpansive mappings and $\mathcal{F} = \cap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (1.1) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $\{\alpha_n^i\} \subseteq [\varepsilon, 1-\varepsilon]$ for all $i = 1, 2, \dots, N$, for some $\varepsilon \in (0,1)$. Then $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for all $i = 1, 2, \dots, N$.*

Proof. Let $x^* \in \mathcal{F} = \cap_{i=1}^N F(T_i)$. Then by Theorem 3.1 (i) and Lemma 2.4, $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Let $\lim_{n\to\infty} \|x_n - x^*\| = a$. If $a = 0$, then by the continuity of each T_i the conclusion follows. Now suppose that $a > 0$. Firstly, we now show that $\lim_{n\to\infty} \|T_N^n x_n - x_n\| = 0$. Since $\{x_n\}$ and $\{u_n^i\}$ are bounded for all $i = 1, 2, \dots, N$, there exists $R > 0$ such that $x_n - x^* + \gamma_n^i(u_n^i - x_n), T_i^n x_n^{i-1} - x^* + \gamma_n^i(u_n^i - x_n) \in B_R(0)$ for all $n \geq 1$ and for all $i = 1, 2, \dots, N$. Using Lemma 2.5, we have

$$\begin{aligned}
\|x_n^N - x^*\|^2 &= \|\alpha_n^N T_N^n x_n^{N-1} + \beta_n^N x_n + \gamma_n^N u_n^N - x^*\|^2 \\
&= \|\alpha_n^N (T_N^n x_n^{N-1} - x^* + \gamma_n^N (u_n^N - x_n)) \\
&\qquad + (1 - \alpha_n^N)(x_n - x^* + \gamma_n^N (u_n^N - x_n))\|^2 \\
&\leq \alpha_n^N |T_N^n x_n^{N-1} - x^* + \gamma_n^N (u_n^N - x_n)\|^2 + (1 - \alpha_n^N) \\
&\qquad \|x_n - x^* + \gamma_n^N (u_n^N - x_n)\|^2 - W_2(\alpha_n^N) g(\|T_N^n x_n^{N-1} - x_n\|) \\
&\leq \alpha_n^N (\|T_N^n x_n^{N-1} - x^*\| + \gamma_n^N \|u_n^N - x_n\|)^2 + (1 - \alpha_n^N) \\
&\qquad (\|x_n - x^*\| + \gamma_n^N \|u_n^N - x_n\|)^2 - W_2(\alpha_n^N) g(\|T_N^n x_n^{N-1} - x_n\|) \\
&\leq \alpha_n^N [(1 + b_n^{N-2})\|x_n - x^*\| + d_n^{N-2} + \gamma_n^N \|u_n^N - x_n\|]^2 + (1 - \alpha_n^N) \\
&\qquad [(1 + b_n^{N-2})\|x_n - x^*\| + d_n^{N-2} + \gamma_n^N \|u_n^N - x_n\|]^2 \\
&\qquad - W_2(\alpha_n^N) g(\|T_N^n x_n^{N-1} - x_n\|) \\
&\leq [(1 + b_n^{N-2})\|x_n - x^*\| + d_n^{N-2} + \gamma_n^N \|u_n^N - x_n\|]^2 \\
&\qquad - W_2(\alpha_n^N) g(\|T_N^n x_n^{N-1} - x_n\|) \\
&\leq [\|x_n - x^*\| + \lambda_n^{N-2}]^2 - W_2(\alpha_n^N) g(\|T_N^n x_n^{N-1} - x_n\|), \qquad (3.1)
\end{aligned}$$

where $\lambda_n^{N-2} = d_n^{N-2} + \gamma_n^N \|u_n^N - x_n\|$. Observe that $\varepsilon^3 \leq W_2(\alpha_n^N)$. Now (3.1) implies that $\varepsilon^3 g(\|T_N^n x_n^{N-1} - x_n\|) \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \rho_n^{N-2}$, where $\rho_n^{N-2} = 2\lambda_n^{N-2} + (\lambda_n^{N-2})^2$. Since $\sum_{n=1}^{\infty} d_n^{N-2} < \infty$ and $\sum_{n=1}^{\infty} \gamma_n^{N-2} < \infty$, we get $\sum_{n=1}^{\infty} \rho_n^{N-2} < \infty$. This implies that $\lim_{n\to\infty} g(\|T_N^n x_n^{N-1} - x_n\|) = 0$. Since g is strictly increasing and continuous at 0, it follows that $\lim_{n\to\infty} \|T_N^n x_n^{N-1} - x_n\| = 0$. Since for all N, T_N is asymptotically nonexpansive,

$$\|x_n - x^*\| \leq \|x_n - T_N^n x_n^{N-1}\| + \|T_N^n x_n^{N-1} - x^*\|$$

$$= \|x_n - T_N^n x_n^{N-1}\| + (1 + r_n^N)\|x_n^{N-1} - x^*\|$$

for all $n \geq 1$. Thus

$$a = \lim_{n\to\infty} \|x_n - x^*\| \leq \liminf_{n\to\infty} \|x_n^{N-1} - x^*\| \leq \limsup_{n\to\infty} \|x_n^{N-1} - x^*\| \leq a$$

and therefore $\lim_{n\to\infty} \|x_n^{N-1} - x^*\| = a$. Using the same argument in the proof above, we have

$$\begin{aligned}
\|x_n^{N-1} - x^*\|^2 \leq\ & \alpha_n^{N-1}\|T_{N-1}^n x_n^{N-2} - x^* + \gamma_n^{N-1}(u_n^{N-1} - x_n)\|^2 + (1 - \alpha_n^{N-1}) \\
& \|x_n - x^* + \gamma_n^{N-1}(u_n^{N-1} - x_n)\|^2 - W_2(\alpha_n^{N-1})g(\|T_{N-1}^n x_n^{N-2} - x_n\|) \\
\leq\ & \alpha_n^{N-1}[(1 + b_n^{N-3})\|x_n - x^*\| + d_n^{N-3} + \gamma_n^{N-1}\|u_n^{N-1} - x_n\|]^2 \\
& + (1 - \alpha_n^{N-1})[(1 + b_n^{N-3})\|x_n - x^*\| + d_n^{N-3} + \gamma_n^{N-1}\|u_n^{N-1} - x_n]^2 \\
& - W_2(\alpha_n^{N-1})g(\|T_{N-1}^n x_n^{N-2} - x_n\|) \\
\leq\ & b_n^{N-3})\|x_n - x^*\| + d_n^{N-3} + \gamma_n^{N-1}\|u_n^{N-1} - x_n\|]^2 \\
& - W_2(\alpha_n^{N-1})g(\|T_{N-1}^n x_n^{N-2} - x_n\|) \\
\leq\ & [\|x_n - x^*\| + \lambda_n^{N-3}]^2 - W_2(\alpha_n^{N-1})g(\|T_{N-1}^n x_n^{N-2} - x_n\|), \qquad (3.2)
\end{aligned}$$

where $\lambda_n^{N-3} = d_n^{N-3} + \gamma_n^{N-1}\|u_n^{N-1} - x_n\|$. This implies that

$$\varepsilon^3 g(\|T_{N-1}^n x_n^{N-2} - x_n\|) \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \rho_n^{N-3},$$

where $\rho_n^{N-3} = 2\lambda_n^{N-3} + (\lambda_n^{N-3})^2$ and therefore $\lim_{n\to\infty} \|T_{N-1}^n x_n^{N-2} - x_n\| = 0$. Thus, we have

$$\begin{aligned}
& \|x_n - T_N^n x_n\| \\
\leq & \|x_n - T_N^n x_n^{N-1}\| + \|T_N^n x_n^{N-1} - T_N^n x_n\| \\
\leq & \|x_n - T_N^n x_n^{N-1}\| + (1 + r_n^N)\|x_n^{N-1} - x_n\| \\
\leq & \|x_n - T_N^n x_n^{N-1}\| + (1 + r_n^N)\|\alpha_n^{N-1} T_{N-1}^n x_n^{N-2} + \beta_n^{N-1} x_n + \gamma_n^{N-1} u_n^{N-1} - x_n\| \\
\leq & \|x_n - T_N^n x_n^{N-1}\| + (1 + r_n^N)[\alpha_n^{N-1}\|T_{N-1}^n x_n^{N-2} - x_n\| + \gamma_n^{N-1}\|u_n^{N-1} - x_n\|].
\end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - T_N^n x_n^{N-1}\| = 0$, $\lim_{n\to\infty} \|x_n - T_{N-1}^n x_n^{N-2}\| = 0$ and $\sum_{n=1}^\infty \gamma_n^{N-1} < \infty$, $\sum_{n=1}^\infty r_n^N < \infty$, it follows that $\lim_{n\to\infty} \|x_n - T_N^n x_n\| = 0$. Similarly, by using the same argument as in the proof above we have $\lim_{n\to\infty} \|x_n - T_{N-2}^n x_n^{N-3}\| = \lim_{n\to\infty} \|x_n - T_{N-3}^n x_n^{N-4}\| =, \dots, = \lim_{n\to\infty} \|x_n - T_2^n x_n^1\| = 0$. This implies that

$$\lim_{n\to\infty} \|x_n - T_{N-1}^n x_n\| = \lim_{n\to\infty} \|x_n - T_{N-2}^n x_n\| = \cdots = \lim_{n\to\infty} \|x_n - T_3^n x_n\| = 0.$$

It remains to show that

$$\lim_{n\to\infty} \|x_n - T_1^n x_n\| = 0, \qquad \lim_{n\to\infty} \|x_n - T_2^n x_n\| = 0.$$

Note that

$$\begin{aligned}
\|x_n^1 - x^*\|^2 \leq\ & \alpha_n^1(\|T_1^n x_n - x^*\| + \gamma_n^1\|u_n^1 - x^*\|)^2 + (1 - \alpha_n^1) \\
& (\|x_n - x^*\| + \gamma_n^1\|u_n^1 - x^*\|)^2 - W_2(\alpha_n^1)g(\|T_1^n x_n - x_n\|)
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n^1[(1+r_n^1)\|x_n - x^*\| + \gamma_n^1\|u_n^1 - x^*\|]2 + (1-\alpha_n^1) \\
&\quad [(1+r_n^1)\|x_n - x^*\| + \gamma_n^1\|u_n^1 - x^*\|]^2 - W_2(\alpha_n^1)g(\|T_1^n x_n - x_n\|) \\
&\leq [(1+r_n^1)\|x_n - x^*\| + \gamma_n^1\|u_n^1 - x^*\|]^2 \\
&\quad - W_2(\alpha_n^1)g(\|T_1^n x_n - x_n\|) \\
&\leq [\|x_n - x^*\| + \gamma_n^1\|u_n^1 - x^*\|]^2 - W_2(\alpha_n^1)g(\|T_1^n x_n - x_n\|).
\end{aligned}$$

Thus, we have $\varepsilon^3 g(\|T_1^n x_n - x_n\|) \leq [\|x_n - x^*\| + \gamma_n^1\|u_n^1 - x^*\|]^2 - \|x_n^1 - x^*\|^2$ and therefore $\lim_{n\to\infty}\|T_1^n x_n - x_n\| = 0$. Since

$$\begin{aligned}
\|x_n - T_2^n x_n\| &\leq \|x_n - T_2^n x_n^1\| + \|T_2^n x_n^1 - T_2^n x_n\| \\
&\leq \|x_n - T_2^n x_n^1\| + (1+r_n^2)\|x_n^1 - x_n\| \\
&\leq \|x_n - T_2^n x_n^1\| + (1+r_n^2)\|\alpha_n^1 T_1^n x_n + \beta_n^1 x_n + \gamma_n^1 u_n^1 - x_n\| \\
&\leq \|x_n - T_2^n x_n^1\| + (1+r_n^2)[\alpha_n^1\|T_1^n x_n - x_n\| + \gamma_n^1\|u_n^1 - x_n\|],
\end{aligned}$$

it follows that $\lim_{n\to\infty}\|T_2^n x_n - x_n\| = 0$. Therefore $\lim_{n\to\infty}\|T_i^n x_n - x_n\| = 0$ for all $i = 1, 2, \ldots, N$. On the other hand, by Remark 2.8, it is clear that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Therefore, by Lemma 2.7, we can conclude that $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$ for all $i = 1, 2, \ldots, N$. This completes the proof. □

Theorem 3.3. *Let E be a real uniformly convex Banach space satisfying Opial's condition and K be a nonempty closed convex subset of E. Let $T_1, T_2, \ldots, T_N \colon K \to K$ be N asymptotically nonexpansive mappings and $\mathcal{F} = \cap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by* (1.1)*. Then $\{x_n\}$ converges weakly to a common fixed point of the mappings $\{T_1, T_2, \ldots, T_N\}$.*

Proof. From Theorem 3.2, we have $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$. It remains to show that $\{x_n\}$ has a unique weak subsequential limit in $\mathcal{F} = \cap_{i=1}^N F(T_i)$. To prove this, let u and v be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ respectively. Since $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$ and $I - T_i$ is demiclosed with respect to zero for all $i = 1, 2, \ldots, N$ by Lemma 2.6, we obtain that $T_i u = u$ for all $i = 1, 2, \ldots, N$. Similarly, we can prove that $v \in \mathcal{F} = \cap_{i=1}^N F(T_i)$. Now we prove that $u = v$. If $u \neq v$, then by Opial's condition

$$\begin{aligned}
\lim_{n\to\infty}\|x_n - u\| &= \lim_{n_i\to\infty}\|x_{n_i} - u\| \\
&< \lim_{n_i\to\infty}\|x_{n_i} - v\| \\
&= \lim_{n\to\infty}\|x_n - v\| \\
&< \lim_{n_j\to\infty}\|x_{n_j} - u\| \\
&= \lim_{n\to\infty}\|x_n - u\|
\end{aligned}$$

which is a contradiction. Therefore, we have $u = v$. This completes the proof. □

Remark 3.4. Theorem 3.3 extends Theorem 1 of Khan and Takahashi [13] and Theorem 2.1 of Schu [22] to the case of multi step iteration and finite families of asymptotically nonexpansive mappings and relaxed the condition of boundedness on K.

It is well known that every continuous and demicompact mapping must satisfy condition (A) [24]. Since every completely continuous mapping $T\colon K \to K$ is continuous and demicompact, it satisfies condition (A). Therefore to study strong convergence of $\{x_n\}$ defined by (1.1), we use condition (B) instead of the complete continuity of mappings $T_1, T_2, \dots, T_N$.

Now we shall prove the following strong convergence theorem by using condition (B):

Theorem 3.5. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $T_1, T_2, \dots, T_N\colon K \to K$ be N asymptotically nonexpansive mappings and $\mathcal{F} = \cap_{i=1}^N F(T_i) \neq \emptyset$. Suppose $\{T_1, T_2, \dots, T_N\}$ satisfies condition (B). Let $\{x_n\}$ be the sequence defined by (1.1) with $\sum_{n=1}^\infty \gamma_n^i < \infty$ and $\{\alpha_n^i\} \subseteq [\varepsilon, 1-\varepsilon]$ for all $i = 1, 2, \dots, N$, for some $\varepsilon \in (0, 1)$. Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \dots, T_N\}$.*

Proof. From Theorem 3.1(i) and by Lemma 2.4, we see that $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in \mathcal{F} = \cap_{i=1}^N F(T_i)$. Let $\lim_{n\to\infty} \|x_n - x^*\| = a$ for some $a \geq 0$. Without loss of generality, suppose $a = 0$, then there is nothing to prove. If we assume $a > 0$, we have from the proof of Theorem 3.1(i)

$$\|x_{n+1} - x^*\| \leq (1 + b_n^{N-1})\|x_n - x^*\| + d_n^{N-1}, \quad \forall n \in N$$

where $\{b_n^i\}_{n=1}^\infty$ and $\{d_n^i\}_{n=1}^\infty$ for all $i = 1, 2, \dots, N$ are nonnegative real sequences such that $\sum_{n=1}^\infty b_n^i < \infty$ and $\sum_{n=1}^\infty d_n^i < \infty$ for all $i = 1, 2, \dots, N$. This gives that

$$d(x_{n+1}, \mathcal{F}) \leq (1 + b_n^{N-1}) d(x_n, \mathcal{F}) + d_n^{N-1}, \quad \forall n \in N.$$

Applying Lemma 2.4 to the above inequality, we obtain that $\lim_{n\to\infty} d(x_n, \mathcal{F})$ exists. Also by Theorem 3.2, $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for all $i = 1, 2, \dots, N$. Since $\{T_1, T_2, \dots, T_N\}$ satisfies condition (B), we conclude that $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$. Next we show that $\{x_n\}$ is a Cauchy sequence. Since $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$, given any $\varepsilon > 0$, there exists a natural number n_0 such that $d(x_n, \mathcal{F}) < \frac{\varepsilon}{3}$ for all $n \geq n_0$. So we can find $p^* \in \mathcal{F}$ such that $\|x_{n_0} - p^*\| < \frac{\varepsilon}{2}$. For all $n \geq n_0$ and $m \geq 1$, we have

$$\begin{aligned}\|x_{n+m} - x_n\| &\leq \|x_{n+m} - p^*\| + \|x_n - p^*\| \\ &\leq \|x_{n_0} - p^*\| + \|x_{n_0} - p^*\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence and so is convergent since E is complete. Let $\lim_{n\to\infty} x_n = q^*$. Then $q^* \in K$. It remains to show that $q^* \in F$. Let $\varepsilon_1 > 0$ be given. Then there exists a natural number n_1 such that $\|x_n - q^*\| < \frac{\varepsilon_1}{4}$ for all $n \geq n_1$. Since $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$, there exists a natural number $n_2 \geq n_1$ such that for all $n \geq n_2$ we have $d(x_n, \mathcal{F}) < \frac{\varepsilon_1}{5}$ and in particular, we have $d(x_{n_2}, \mathcal{F}) < \frac{\varepsilon_1}{5}$. Therefore, there exists $w^* \in \mathcal{F}$ such that $\|x_{n_2} - w^*\| < \frac{\varepsilon_1}{4}$. For any $i \in I$ and $n \geq n_2$, we have

$$\begin{aligned}
\|T_i q^* - q^*\| &\leq \|T_i q^* - w^*\| + \|w^* - q^*\| \\
&\leq 2\|q^* - w^*\| \\
&\leq 2(\|q^* - x_{n_2}\| + \|x_{n_2} - w^*\|) \\
&< 2(\frac{\varepsilon_1}{4} + \frac{\varepsilon_1}{4}) \\
&< \varepsilon_1.
\end{aligned}$$

This implies that $T_i q^* = q^*$. Hence $q^* \in F(T_i)$ for all $i \in I$ and so $q^* \in \mathcal{F} = \cap_{i=1}^N F(T_i)$. This completes the proof. □

Remark 3.6. (1) Theorem 3.5 extend Theorem 2 of Khan and Fukhar-ud-din [12], Theorem 2.4 of Plubtieng et al. [20], Theorem 2 and 3 of Rhoades [21], Theorem 1.5 of Schu [22] and Theorems 2.1-2.3 of Xu and Noor [28] to the case of multi-step iteration and finite families of asymptotically nonexpansive mappings and relaxed the condition of boundedness on K. Also our iteration scheme generalizes the iteration scheme of Noor [28].

(2) Theorem 3.5 also generalizes Theorem 3.5 of Chidume and Ali [4] to the case of the iteration with errors in the sense of Xu [30].

For our next result, we shall need the following definition:

Definition 3.7. Let C be a nonempty closed subset of a Banach space E. A mapping $T\colon C \to C$ is said to be semi-compact, if for any bounded sequence $\{x_n\}$ in C such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\lim_{i\to\infty} x_{n_i} = x \in C$.

Theorem 3.8. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $T_1, T_2, \ldots, T_N\colon K \to K$ be N asymptotically nonexpansive mappings and $\mathcal{F} = \cap_{i=1}^N F(T_i) \neq \emptyset$. Suppose that one of the mappings in $\{T_1, T_2, \ldots, T_N\}$ is semi-compact. Let $\{x_n\}$ be the sequence defined by (1.1) with $\sum_{n=1}^\infty \gamma_n^i < \infty$ and $\{\alpha_n^i\} \subseteq [\varepsilon, 1-\varepsilon]$ for all $i = 1, 2, \ldots, N$, for some $\varepsilon \in (0, 1)$. Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \ldots, T_N\}$.*

Proof. Suppose that T_{i_0} is semi-compact for some $i_0 \in \{1, 2, \ldots, N\}$. By Theorem 3.2, we have

$$\lim_{n\to\infty} \|x_n - T_{i_0} x_n\| = 0. \tag{3.3}$$

So there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{n_j\to\infty} x_{n_j} = x^* \in K$. So from (3.3), we have $\lim_{n_j\to\infty} \|x_{n_j} - T_j x_{n_j}\| = 0$ for all $j \in \{1, 2, \ldots, N\}$ and so $\|x^* - T_j x^*\| = 0$ for all $j \in \{1, 2, \ldots, N\}$. This implies that $x^* \in \mathcal{F} = \cap_{i=1}^N F(T_i)$. Since $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$, it follows, as in the proof of Theorem 3.5, that $\{x_n\}$ converges strongly to some common fixed point in $\mathcal{F}$. This completes the proof. □

Remark 3.9. Theorem 3.8 extends Theorem 2 of Osilike and Aniagbosor [18] and Theorem 2.2 of Schu [22] to the case of finite families of asymptotically nonexpansive mappings and multi-step iteration and relaxed the condition of boundedness on K.

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