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ON THE RATIONAL RECURSIVE TWO SEQUENCES $x_{n+1} = ax_{n-k} + bx_{n-k}/\left(cx_n + \delta dx_{n-k}\right)$

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Abstract. The main objective of this paper is to study some qualitative behavior of the solutions of the two difference equations

$$x_{n+1} = ax_{n-k} + bx_{n-k}/\left(cx_n + \delta dx_{n-k}\right), \quad n = 0, 1, 2, \ldots,$$

where the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ are arbitrary positive real numbers and the coefficients a, b, c and d are positive constants, while k is a positive integer number and $\delta = \pm 1$. Some numerical examples are given to illustrate our results.

1. Introduction

The qualitative study of difference equations is a fertile research area and increasingly attracts many mathematicians. This topic draws its importance from the fact that many real life phenomena are modeled using difference equations. Examples from economy, biology, etc. can be found in [2, 16, 19, 28]. It is known that nonlinear difference equations are capable of producing a complicated behavior regardless its order. This can be easily seen from the family $x_{n+1} = g_\mu(x_n)$, $\mu > 0$, $n \geq 0$. This behavior is ranging according to the value of μ, from the existence of a bounded number of periodic solutions to chaos.

There has been a great interest in studying the global attractivity, the boundedness character and the periodicity nature of nonlinear difference equations. For example, in the articles [9, 29–31] closely related global convergence results were obtained which can be applied to nonlinear difference equations in proving that every solution of these equations converges to a period two solution. For other closely related results, see [3–5, 11, 14, 15] and the references cited therein. The study of these equations is challenging and rewarding and is still in its infancy. We believe that the nonlinear rational difference equations are of paramount importance in their own right. Furthermore the results about such equations offer prototypes for the development of the basic theory of the global behavior of nonlinear difference equations.

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Our goal in this paper is to investigate some qualitative behavior of the solutions of the two difference equations

$$x_{n+1} = ax_{n-k} + \frac{bx_{n-k}}{cx_n - dx_{n-k}}, \qquad n = 0, 1, 2, \ldots \tag{1.1}$$

and

$$x_{n+1} = ax_{n-k} + \frac{bx_{n-k}}{cx_n + dx_{n-k}}, \qquad n = 0, 1, 2, \ldots, \tag{1.2}$$

where the initial conditions x_{-k},..., x_{-1}, x_0 are arbitrary positive real numbers and the coefficients a, b, c and d are positive constants, while k is a positive integer number. The case where any of a, c, d is allowed to be zero gives different special cases of the two difference equations (1.1) and (1.2) which are studied by many authors, (see for example [3, 9, 12, 14, 18, 31]). For the related work see [1, 2, 4, 5, 7, 9-11, 13-15, 17, 19-30, 32–40]. Note that Eqs. (1.1) and (1.2) can be considered as a generalization of that obtained in [8, 36].

Definition 1. A difference equation of order $(k+1)$ is of the form

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-k}), \qquad n = 0, 1, 2, \ldots, \tag{1.3}$$

where F is a continuous function which maps some set J^{k+1} into J and J is a set of real numbers. An equilibrium point $\widetilde{x}$ of this equation is a point that satisfies the condition $\widetilde{x} = F(\widetilde{x}, \widetilde{x}, \ldots, \widetilde{x})$. That is, the constant sequence $\{x_n\}_{n=-k}^{\infty}$ with $x_n = \widetilde{x}$ *for all* $n \geq -k$ is a solution of that equation.

Definition 2. Let $\widetilde{x} \in (0, \infty)$ be an equilibrium point of the difference equation (1.3). Then

(i) An equilibrium point $\widetilde{x}$ of the difference equation (1.3) is called locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $x_{-k}, \ldots, x_{-1}, x_0 \in (0, \infty)$ with $|x_{-k} - \widetilde{x}| + \ldots + |x_{-1} - \widetilde{x}| + |x_0 - \widetilde{x}| < \delta$, then $|x_n - \widetilde{x}| < \varepsilon$ for all $n \geq -k$.

(ii) An equilibrium point $\widetilde{x}$ of the difference equation (1.3) is called locally asymptotically stable if it is locally stable and there exists $\gamma > 0$ such that, if $x_{-k}, \ldots, x_{-1}, x_0 \in (0, \infty)$ with $|x_{-k} - \widetilde{x}| + \ldots + |x_{-1} - \widetilde{x}| + |x_0 - \widetilde{x}| < \gamma$, then

$$\lim_{n \to \infty} x_n = \widetilde{x}.$$

(iii) An equilibrium point $\widetilde{x}$ of the difference equation (1.3) is called a global attractor if for every $x_{-k}, \ldots, x_{-1}, x_0 \in (0, \infty)$ we have

$$\lim_{n \to \infty} x_n = \widetilde{x}.$$

(iv) An equilibrium point $\widetilde{x}$ of the equation (1.3) is called globally asymptotically stable if it is locally stable and a global attractor.

(v) An equilibrium point $\widetilde{x}$ of the difference equation (1.3) is called unstable if it is not locally stable.

Definition 3. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with prime period p if p is the smallest positive integer having this property.

The linearized equation of the difference equation (1.3) about the equilibrium point $\widetilde{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\widetilde{x}, \widetilde{x}, \ldots, \widetilde{x})}{\partial x_{n-i}} y_{n-i}. \tag{1.4}$$

Now assume that the characteristic equation associated with (1.4) is

$$p(\lambda) = p_0 \lambda^k + p_1 \lambda^{k-1} + \ldots + p_{k-1}\lambda + p_k = 0, \tag{1.5}$$

where $p_i = \partial F(\widetilde{x}, \widetilde{x}, \ldots, \widetilde{x}) / \partial x_{n-i}$.

Theorem 1. [19]. *Assume that $p_i \in R$, $i = 1, 2, \ldots,$ and $k \in \{0, 1, 2, \ldots\}$. Then*

$$\sum_{i=1}^{k} |p_i| < 1 \tag{1.6}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + \ldots + p_k x_n = 0, \quad n = 0, 1, 2, \ldots \tag{1.7}$$

Theorem 2 (See [15, 19, 20] The linearized stability theorem)**.** *Suppose F is a continuously differentiable function defined on an open neighborhood of the equilibrium $\widetilde{x}$. Then the following statements are true.*

(i) *If all roots of the characteristic equation* (1.5) *of the linearized equation* (1.4) *have absolute value less than one, then the equilibrium point $\widetilde{x}$ is locally asymptotically stable.*

(ii) *If at least one root of equation* (1.5) *has absolute value greater than one, then the equilibrium point $\widetilde{x}$ is unstable.*

The following theorem will be useful for the proof of our main results in this paper.

Theorem 3 (See [15, p. 18])**.** *Let $F : [a, b]^{k+1} \longrightarrow [a, b]$ be a continuous function, where k is a positive integer, and where $[a, b]$ is an interval of real numbers and consider the difference equation* (1.3). *Suppose that F satisfies the following conditions:*

(i) *For each integer i with $1 \leq i \leq k + 1$, the function $F(z_1, z_2, ..., z_{k+1})$ is weakly monotonic in z_i for fixed $z_1, z_2, ..., z_{i-1}, z_{i+1}, \ldots, z_{k+1}$.*

(ii) *If (m, M) is a solution of the system*

$$m = F(m_1, m_2, \ldots, m_{k+1}) \quad \text{and} \quad M = F(M_1, M_2, \ldots, M_{k+1}),$$

then $m = M$, where for each $i = 1, 2, \ldots, k + 1$, we set

$$m_i = \begin{cases} m & if \;\; F \;\; nondecreasing \;\; in \;\; z_i, \\ M & if \;\; F \;\; nonincreasing \;\; in \;\; z_i, \end{cases}$$

and

$$M_i = \begin{cases} M & if \;\; F \;\; nondecreasing \;\; in \;\; z_i, \\ \\ m & if \;\; F \;\; nonincreasing \;\; in \;\; z_i. \end{cases}$$

Then there exists exactly one equilibrium point $\widetilde{x}$ *of the difference equation* (1.3), *and every solution of* (1.3) *converges to* $\widetilde{x}$.

2. Periodic solutions of equation (1.1)

Theorem 4. *If* k *is an even positive integer and* $c \neq d$, *then equation* (1.1) *has no positive solutions of prime period two.*

Proof. Assume that there exists distinctive positive solution

$$\ldots, P, Q, P, Q, \ldots$$

of prime period two of the difference equation (1.1).

If k is an even positive integer, then $x_n = x_{n-k}$. It follows from equation (1.1) that

$$P = aQ + \frac{bQ}{cQ - dQ} \quad \text{and} \quad Q = aP + \frac{bP}{cP - dP}, \tag{2.1}$$

provided that $c \neq d$. Hence we deduce from (2.1) that $(P - Q)(a + 1) = 0$.Thus $P = Q$. This is a contradiction. Therefore, the proof of Theorem 4 is complete. □

Theorem 5. *If* k *is an odd positive integer and* $a \neq 1$, *then equation* (1.1) *has no positive solutions of prime period two.*

Proof. Assume that there exists distinctive positive solution

$$\ldots, P, Q, P, Q, \ldots$$

of prime period two of the difference equation (1.1).

If k is an odd positive integer, then $x_{n+1} = x_{n-k}$. It follows from the difference equation (1.1) that

$$P = aP + \frac{bP}{cQ - dP} \quad \text{and} \quad Q = aQ + \frac{bQ}{cP - dQ}.$$

Consequently, we obtain

$$cPQ - dP^2 = acPQ - adP^2 + bP, \tag{2.2}$$

and

$$cPQ - dQ^2 = acPQ - adQ^2 + bQ. \tag{2.3}$$

By adding (2.2) and (2.3) we deduce after some reduction that

$$2(1 - a)(c + d)PQ = 0.$$

Since $a \neq 1$, then

$$PQ = 0. \tag{2.4}$$

Since P, Q are both positive, then we have a contradiction. Therefore, the proof of Theorem 5 is now complete. □

3. The stability of the equilibrium point of equation (1.1)

In this section we study the local stability character of the solutions of the difference equation (1.1). The equilibrium points of the difference equation (1.1) are given by the relation

$$\widetilde{x} = a\widetilde{x} + \frac{b\widetilde{x}}{c\widetilde{x} - d\widetilde{x}}. \tag{3.1}$$

If $(a-1)(d-c) > 0$, then the only positive equilibrium point $\widetilde{x}$ of the difference equation (1.1) is given by

$$\widetilde{x} = \frac{b}{(a-1)(d-c)}. \tag{3.2}$$

Let $F : (0,\infty)^2 \longrightarrow (0,\infty)$ be a continuous function defined by

$$F(u_0, u_1) = au_1 + \frac{bu_1}{cu_0 - du_1}, \tag{3.3}$$

provided that $cu_0 \neq du_1$. Therefore,

$$\frac{\partial F(u_0,u_1)}{\partial u_0} = -\frac{bcu_1}{(cu_0 - du_1)^2} \quad \text{and} \quad \frac{\partial F(u_0,u_1)}{\partial u_1} = a + \frac{bcu_0}{(cu_0 - du_1)^2}. \tag{3.4}$$

Then we see that

$$\frac{\partial F(\widetilde{x},\widetilde{x})}{\partial u_0} = -\frac{c(a-1)}{(d-c)} = \rho_0 \quad \text{and} \quad \frac{\partial F(\widetilde{x},\widetilde{x})}{\partial u_1} = a + \frac{c(a-1)}{(d-c)} = \rho_1, \tag{3.5}$$

provided that $d \neq c$. Then the linearized equation of the difference equation (1.1) about $\widetilde{x}$ is

$$y_{n+1} - \rho_0\, y_n - \rho_1\, y_{n-k} = 0. \tag{3.6}$$

Theorem 6. *Assume that* $a \neq 1$, $d \neq c$ *and*

$$|c - ac| + |ad - c| < |d - c|, \tag{3.7}$$

Then the equilibrium point $\widetilde{x}$ *of the difference equation* (1.1) *is locally asymptotically stable.*

Proof. From (3.5) we deduce for $d \neq c$ and $a \neq 1$ that

$$\begin{aligned} |\rho_0| + |\rho_1| &= \left| -\frac{c(a-1)}{(d-c)} \right| + \left| a + \frac{c(a-1)}{(d-c)} \right| \\ &= \frac{|c-ac|}{|d-c|} + \frac{|ad-c|}{|d-c|}. \end{aligned} \tag{3.8}$$

From (3.7) and (3.8), we deduce that

$$|\rho_0| + |\rho_1| < 1. \tag{3.9}$$

It follows by Theorems 1, 2 that equation (1.1) is locally asymptotically stable. Thus, the proof of Theorem 6 is complete. ☐

4. Global attractor of the equilibrium point of equation (1.1)

In this section we investigate the global attractivity character of the solutions of the difference equation (1.1).

Theorem 7. *The equilibrium point* $\widetilde{x}$ *of the difference equation* (1.1) *is a global attractor if* $a \neq 1$.

Proof. By using (3.4), we can see that the function $F(u_0, u_1)$ which is defined by (3.3) is decreasing in u_0 and increasing in u_1. Suppose that (m, M) is a solution of the system

$$m = F(M, m) \quad \text{and} \quad M = F(m, M). \tag{4.1}$$

Then we get

$$m = F(M, m) = am + \frac{bm}{cM - dm},$$

and

$$M = F(m, M) = aM + \frac{bM}{cm - dM}.$$

Consequently, we have

$$\frac{1}{cM - dm} = \frac{1}{cm - dM} = (1 - a)/b. \tag{4.2}$$

Since $a \neq 1$, we deduce from (4.2) that $M = m$. It follows by Theorem 3 that $\widetilde{x}$ is a global attractor of the difference equation (1.1). Thus, the proof of Theorem 7 is complete. ☐

5. Periodic solutions of equation (1.2)

Theorem 8. *If* k *is an even positive integer, then equation* (1.2) *has no positive solutions of prime period two.*

Proof. Assume that there exists distinctive positive solution

$$\ldots, P, Q, P, Q, \ldots$$

of prime period two of the difference equation (1.2).

If k is an even positive integer, then $x_n = x_{n-k}$. It follows from equation (1.2) that

$$P = aQ + \frac{bQ}{cQ + dQ} \quad \text{and} \quad Q = aP + \frac{bP}{cP + dP}. \tag{5.1}$$

Consequently, we deduce from (5.1) that

$$(P - Q)(a + 1) = 0. \tag{5.2}$$

Then, we have $P = Q$. This is a contradiction. Hence, the proof of Theorem 8 is complete. ☐

Theorem 9. *If k is an odd positive integer, then the neceessary and insufficient condition for equation* (1.2) *to have positive solutions of prime period two is that*

$$c > 5d, \tag{5.3}$$

provided that $0 < a < 1$ *and* $c > d$.

Proof. Assume that there exists distinctive positive solution

$$\ldots, P, Q, P, Q, \ldots$$

of prime period two of the difference equation (1.2).

If k is an odd positive integer, then $x_{n+1} = x_{n-k}$. It follows from the difference equation (1.2) that

$$P = aP + \frac{bP}{cQ + dP} \quad \text{and} \quad Q = aQ + \frac{bQ}{cP + dQ}.$$

Consequently, we have

$$cPQ + dP^2 = acPQ - adP^2 + bP, \tag{5.4}$$

and

$$cPQ + dQ^2 = acPQ - adQ^2 + bQ. \tag{5.5}$$

By subtracting (5.4) from (5.5), we deduce that

$$P + Q = \frac{b}{d\,(1-a)}, \tag{5.6}$$

while, by adding (5.4) and (5.5), we have

$$PQ = \frac{b^2}{d\,(1-a)^2\,(c-d)}, \tag{5.7}$$

where $0 < a < 1$ and $c > d$. Assume that P and Q are two positive distinct real roots of the quadratic equation

$$t^2 - (\ P + Q)\,t + PQ = 0. \tag{5.8}$$

Thus, we deduce that

$$\left(\frac{b}{d\,(1-a)}\right)^2 > \frac{4b^2}{d\,(1-a)^2\,(c-d)}. \tag{5.9}$$

From (5.9), we obtain the condition (5.3). Thus, the necessary condition is satisfied. Conversely, suppose that the condition (5.3) is valid. Then, we deduce immediately from (5.3) that the inequality (5.9) holds. Consequently, there exist two positive distinct real numbers P and Q such that

$$P = \frac{b+\beta}{2d\,(1-a)} \quad \text{and} \quad Q = \frac{b-\beta}{2d\,(1-a)}\ , \tag{5.10}$$

where $\beta = \sqrt{b^2 - 4b^2 d/\,(c-d)}$. Thus, P and Q represent two positive distinct real roots of the quadratic equation (5.8). Now, we are going to prove that P

and Q are not positive solutions of prime period two of the difference equation (1.2). To this end, we assume that

$$x_{-k} = P, \quad x_{-k+1} = Q, \ldots, \quad x_{-1} = P, \quad \text{and} \quad x_0 = Q.$$

We shall show that $x_1 \neq P$. To this end, we deduce from the difference equation (1.2) that

$$x_1 = ax_{-k} + \frac{bx_{-k}}{cx_0 + dx_{-k}} = aP + \frac{bP}{cQ + dP}. \tag{5.11}$$

Thus, we deduce from (5.10) and (5.11) that

$$\begin{aligned} x_1 - P &= \frac{c(a-1)PQ + d(a-1)P^2 + bP}{cQ + dP} \\ &= \frac{c(a-1)\left[\frac{b^2-\beta^2}{4d^2(1-a)^2}\right] + d(a-1)\left[\frac{b+\beta}{2d(1-a)}\right]^2 + b\left[\frac{b+\beta}{2d(1-a)}\right]}{c\left[\frac{b-\beta}{2d(1-a)}\right] + d\left[\frac{b+\beta}{2d(1-a)}\right]}. \end{aligned} \tag{5.12}$$

Multiplying the denominator and numerator of (5.12) by $4d^2(1-a)^2$ we get

$$\begin{aligned} x_1 - P &= \frac{2bd(b+\beta) - c\left(b^2 - \beta^2\right) - d(b+\beta)^2}{2d\left[c(b-\beta) + d(b+\beta)\right]} \\ &= \frac{b^2(d-c) + (c-d)\left[b^2 - \frac{4b^2 d}{c-d}\right]}{2d\left[c(b-\beta) + d(b+\beta)\right]} \\ &= \frac{-2b^2}{c(b-\beta) + d(b+\beta)} \neq 0. \end{aligned} \tag{5.13}$$

Thus $x_1 \neq P$. This shows that equation (1.2) has no positive solutions of prime period two. Hence the proof of Theorem 9 is now complete. □

From Theorem 9, we have the following result:

Theorem 10. *If either $a > 1$ or $c < d$ and if both $a > 1$ and $c < d$ hold, then if k is an odd positive integer, then the equation* (1.2) *has no positive solutions of prime period two.*

6. The stability of the equilibrium point of equation (1.2)

In this section we study the local stability character of the solutions of the difference equation (1.2). The equilibrium points of the difference equation (1.2) are given by the relation

$$\widetilde{x} = a\widetilde{x} + \frac{b\widetilde{x}}{c\widetilde{x} + d\widetilde{x}}. \tag{6.1}$$

If $0 < a < 1$, then the only positive equilibrium point $\widetilde{x}$ of the difference equation (1.2) is given by

$$\widetilde{x} = \frac{b}{(1-a)(c+d)}. \tag{6.2}$$

Let $F:(0,\infty)^2 \longrightarrow (0,\infty)$ be a continuous function defined by

$$F(u_0,u_1) = au_1 + \frac{bu_1}{cu_0+du_1}. \tag{6.3}$$

Therefore,

$$\frac{\partial F(u_0,u_1)}{\partial u_0} = -\frac{bcu_1}{(cu_0+du_1)^2} \quad \text{and} \quad \frac{\partial F(u_0,u_1)}{\partial u_1} = a + \frac{bcu_0}{(cu_0+du_1)^2}. \tag{6.4}$$

Then, we see that

$$\frac{\partial F(\widetilde{x},\widetilde{x})}{\partial u_0} = -\frac{c\,(1-a)}{(c+d)} = \rho_0 \quad \text{and} \quad \frac{\partial F(\widetilde{x},\widetilde{x})}{\partial u_1} = a + \frac{c\,(1-a)}{(c+d)} = \rho_1. \tag{6.5}$$

Then, the linearized equation of the difference equation (1.2) about $\widetilde{x}$ is

$$y_{n+1} - \rho_0\, y_n - \rho_1\, y_{n-k} = 0. \tag{6.6}$$

Theorem 11. *Assume that* $0 < a < 1$ *and*

$$c\,(1-a) + ad + c < c + d. \tag{6.7}$$

Then the equilibrium point $\widetilde{x}$ *of the difference equation* (1.2) *is locally asymptotically stable.*

Proof. From (6.5) we deduce for $0 < a < 1$ that

$$\begin{aligned} |\rho_0| + |\rho_1| &= \left| -\frac{c\,(1-a)}{c+d} \right| + \left| a + \frac{c\,(1-a)}{c+d} \right| \\ &= \frac{c\,(1-a)}{c+d} + \frac{ad+c}{c+d}. \end{aligned} \tag{6.8}$$

From (6.7) and (6.8), we have

$$|\rho_0| + |\rho_1| < 1. \tag{6.9}$$

It follows from Theorems 1, 2 that $\widetilde{x}$ of equation (1.2) is locally asymptotically stable. Hence, the proof of Theorem 11 is complete. ☐

7. Global attractor of the equilibrium point of equation (1.2)

In this section we investigate the global attractivity character of the solutions of the difference equation (1.2).

Theorem 12. *The equilibrium point* $\widetilde{x}$ *of the difference equation* (1.2) *is a global attractor if* $0 < a < 1$ *and* $c > d$.

Proof. By using (6.4), we can see that the function $F(u_0,\ u_1)$ which is defined by (6.3) is decreasing in u_0 and increasing in u_1. Suppose that (m, M) is a solution of the system

$$m = F(M,m) \quad \text{and} \quad M = F(m,M). \tag{7.1}$$

Then we get

$$m = F(M, m) = am + \frac{bm}{cM + dm}\ , \tag{7.2}$$

and

$$M = F(m, M) = aM + \frac{bM}{cm + dM}. \tag{7.3}$$

We deduce from (7.2) and (7.3) that

$$\frac{1}{cM + dm} = \frac{1}{cm + dM} = (1 - a)/b. \tag{7.4}$$

Since, $0 < a < 1$, then the relation (7.4) gives $(M - m)(c - d) = 0$. Since, $c > d$, then $M = m$. It follows by Theorem 3 that $\widetilde{x}$ is a global attractor of the difference equation (2). Thus, the proof of Theorem 12 is complete. □

8. Numerical examples

For confirming the results of this paper, we consider numerical examples which represent different types of solutions to equation (1.1) and (1.2).

Example 1. Figure 1 shows that equation (1.1) has no prime period two solution if $k = 2$, $x_{-2} = 1$, $x_{-1} = 2$, $x_0 = 3$, $a = 500$, $b = 5$, $c = 10$, $d = 30$.

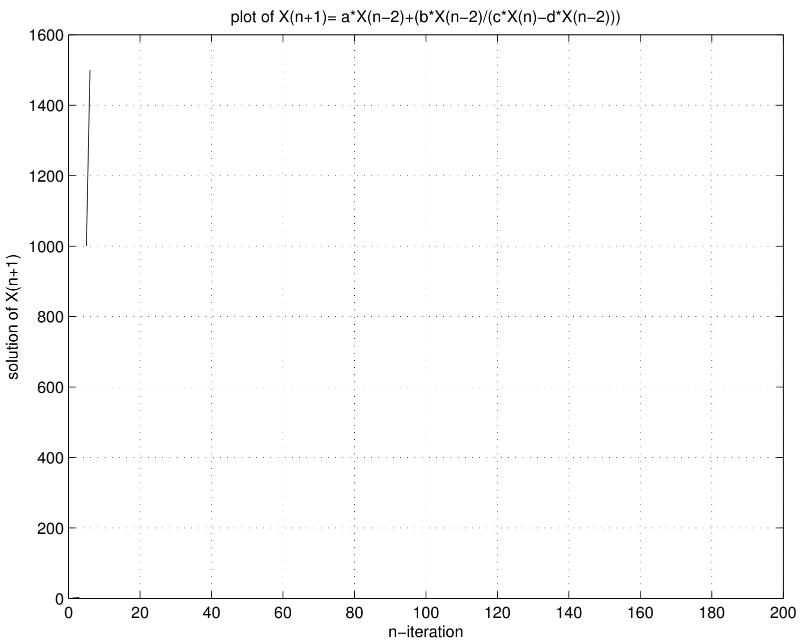

FIGURE 1. $\left(x_{n+1} = 500x_{n-2} + \dfrac{5x_{n-2}}{10x_n - 30x_{n-2}}\right)$

Example 2. Figure 2 shows that Eq. (1.1) has no prime period two solution if $k = 1$, $x_{-1} = 1$, $x_0 = 2$,$a = 2000$, $b = 5$, $c = 100$, $d = 300$.

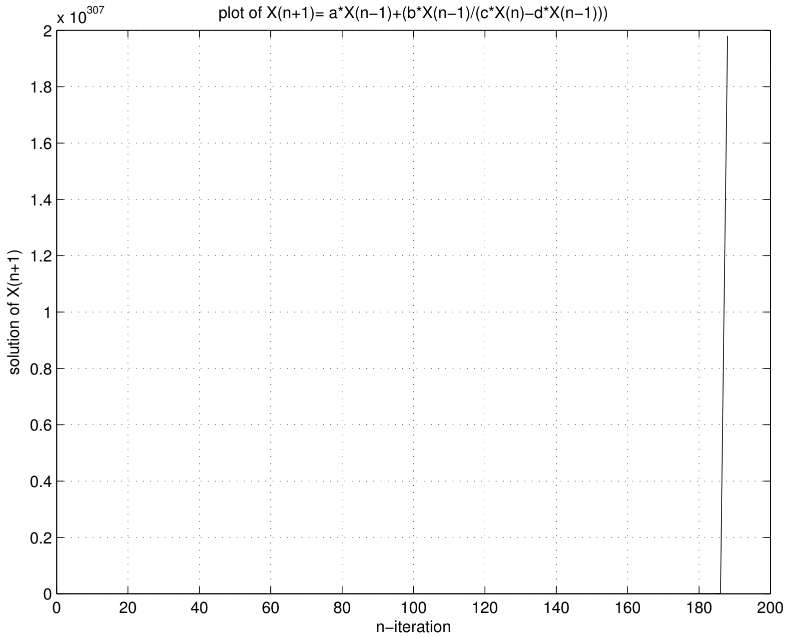

FIGURE 2. $\left(x_{n+1} = 2000x_{n-1} + \dfrac{5x_{n-1}}{100x_n - 300x_{n-1}}\right)$

Example 3. Figure 3 shows that the solution of equation (1.1) is global attractor if $k = 1$, $x_{-1} = 1$, $x_0 = 2$, $a = 0.01$, $b = 5$, $c = 100$, $d = 300$.

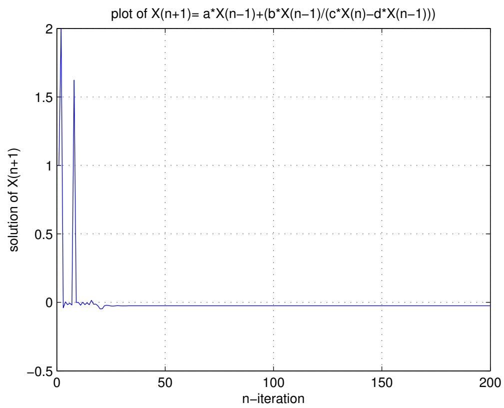

FIGURE 3

Example 4. Figure 4 shows that Eq. (1.2) has no prime period two solution if $k = 2$, $x_{-2} = 1$, $x_{-1} = 2$, $x_0 = 3$, $a = 0.5$, $b = 5$, $c = 300$, $d = 1000$.

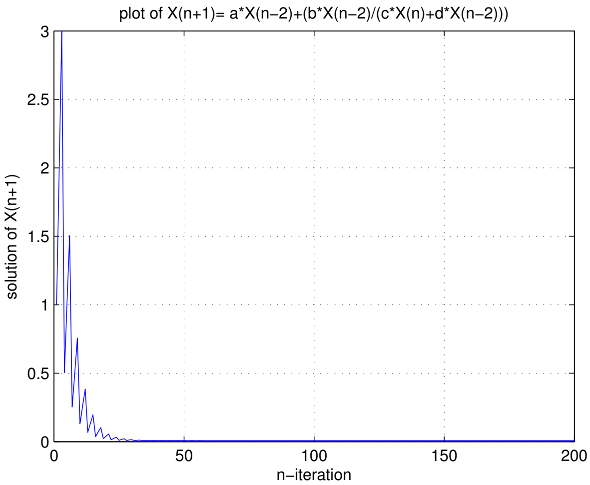

FIGURE 4. $\left(x_{n+1} = 0.5x_{n-2} + \dfrac{5x_{n-2}}{300x_n + 1000x_{n-2}}\right)$

Example 5. Figure 5 shows that equation (1.2) has prime period two solution if $k = 1$, $x_{-1} = 0.036$, $x_0 = 0.96$, $a = 0.5$, $b = 5$, $c = 300$, $d = 10$.

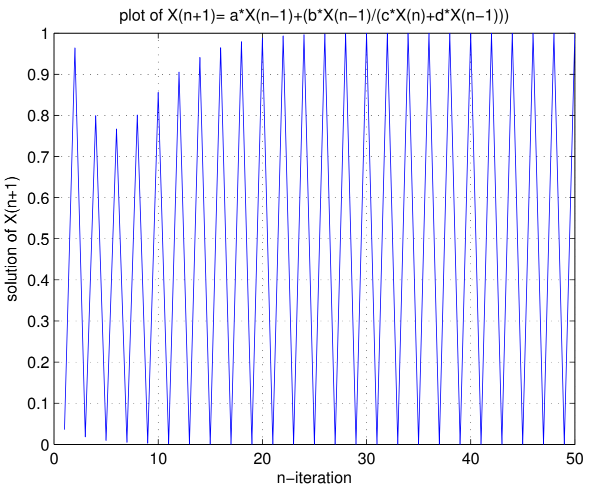

FIGURE 5. $\left(x_{n+1} = 0.5x_{n-1} + \dfrac{5x_{n-1}}{300x_n + 10x_{n-1}}\right)$

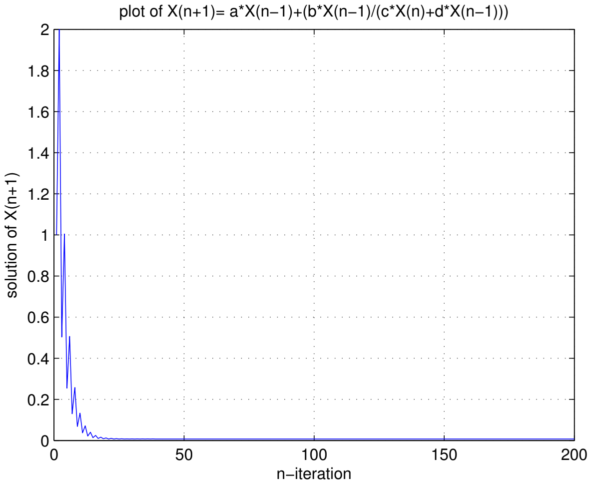

FIGURE 6. $\left(x_{n+1} = 0.5x_{n-1} + \dfrac{5x_{n-1}}{300x_n + 1000x_{n-1}}\right)$

Example 6. Figure 6 shows that the solution of Eq. (1.2) is global stability if $k = 1$, $x_{-1} = 1$, $x_0 = 2$, $a = 0.5$, $b = 5$, $c = 300$, $d = 1000$.

Note that Example 1 verifies Theorem 4 which shows that equation (1.1) has no prime period two solution, while Example 2 verifies Theorem 5 which shows that equation (1.1) has no prime period two solution. But Example 3 verifies Theorem 7 which shows that if $a \neq 1$, then the solution of equation (1.1) is a global attractor. Example 4 verifies Theorem 8 which shows that equation (1.2) has no prime period two solution, while Example 5 verifies Theorem 9 which shows that equation (1.2) has prime period two solution. But Example 6 verifies Theorem 12 which shows that the solution of equation (1.2) is a global attractor.

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