ACTA MATHEMATICA VIETNAMICA
Volume 35, Number 3, 2010, pp. 335–354

RANDOM MATRICES I: COMBINATORIAL PROBLEMS

TRAN VINH LINH AND VAN VU

Abstract. This is the first part of a series of surveys on random matrices. In this part, we focus on problems and results of combinatorial nature.

1. Introduction

The theory of random matrices is a rich topic in mathematics. Beside being interesting in its own right, random matrices play fundamental role in various areas such as statistics, mathematical physics, combinatorics, theoretical computer science, etc. A famous example here is the study of the physicist Wigner, who used the spectrum of random matrices as a model in nuclear physics, and consequently discovered the fundamental semi-circle law.

In the last ten years or so, we have witnessed considerable progress on several long standing problems in random matrix theory. This motivates the current authors for this series of surveys, in which we hope to present the state of the art of the theory and propose a few directions for future research.

In this (the first) part of the series, we focus on problems with combinatorial flavors. These problems usually make sense in the discrete setting, when the random matrix is sampled from a discrete distribution. The most popular models are:

- (Bernoulli) M_n: random matrix of size n whose entries are i.i.d. Bernoulli random variables (taking values ± 1 with probability $1/2$). This is sometimes referred to as the random sign matrix.
- (Symmetric Bernoulli) M_n^{sym}: random symmetric matrix of size n whose (upper triangular) entries are i.i.d. Bernoulli random variables (taking values ± 1 with probability $1/2$).
- (Adjacency matrix of a random graph) Another way to obtain a random matrix is to look at the adjacency matrix of a random graph. This matrix is symmetric and at position ij we write 1 if there is an edge and zero otherwise. Models of random graphs are introduced below.

Received September 16, 2010.
2000 *Mathematics Subject Classification.* 15B52.
Key words and phrases. Random discrete matrices.
This paper is an extended version of an earlier survey [51].

- (Laplacian matrix of a random graph) Instead of the adjacency matrix of a random graph, one can also look at its laplacian. The definition of the laplacian will appear later in the paper.

Model of random graphs. We will focus on two models: Erdös-Rényi and random regular graphs. For more information about these models, we refer to [4, 26].

- (Erdös-Rényi) We denote by $G(n,p)$ a random graph on n vertices, generated by drawing an edge between any two vertices with probability p, independently.
- (Random regular graph) A random regular graph on n vertices with degree d is obtained by sampling uniformly over the set of all simple d-regular graphs on the vertex set $\{1,\dots,n\}$. We denote this graph by $G_{n,d}$.

It is important to notice that the edges of $G_{n,d}$ are not independent. Because of this, this model is usually harder to study, compared to $G(n,p)$.

We denote by $A(n,p)$ ($L(n,p)$) the adjacency (laplacian) matrix of the Erdös-Rényi random graph $G(n,p)$ and by $A_{n,d}$ ($L_{n,d}$) the adjacency (laplacian) matrix of $G_{n,d}$, respectively.

Notation. In the whole paper, we assume that n is large. The asymptotic notation such as o, O, Θ is used under the assumption that $n\to\infty$. We write $A \ll B$ if $A = o(B)$. c denotes a universal constant. All logarithms have natural base, if not specified otherwise.

2. The singular probability

The most famous combinatorial problem concerning random matrices is perhaps the "singularity" problem, which asks for the probability that a random Bernoulli matrix is singular.

Let p_n be the probability that M_n is singular. Notice that a matrix is singular if it has two equal rows and the probability that the first two rows of M_n are equal is 2^{-n}, it follows that

$$p_n \ge 2^{-n}.$$

By choosing any two rows (columns) and also replacing equal by equal up to sign, one can have a slightly better lower bound

$$p_n \ge (4 - o(1))\binom{n}{2}2^{-n} = (\frac{1}{2} + o(1))^n. \tag{2.1}$$

A famous conjecture in the field asserts that this trivial lower bound is sharp.

Conjecture 2.1. $p_n = (1/2 + o(1))^n$.

There are refined versions of the above conjecture (for instance, one can replace $(\frac{1}{2}+o(1))^n$ by $(4+o(1))\binom{n}{2}2^{-n}$, see [27]). However, Conjecture 2.1, as formulated, is still open. The essence of this conjecture is that *the dominating reason for singularity is the dependence between a few rows/columns.*

It is already non-trivial to show that $p_n = o(1)$. It was Komlós [30] who first proved in 1967.

Theorem 2.2.

$$p_n = o(1).$$

In Section 3, we will give a short proof for this fact. The bound on p_n in [30] tends very slowly to zero. Later, Komlós (see [4]) found a new proof which showed $p_n = O(n^{-1/2})$. In 1995, a breakthrough by Kahn, Komlós and Szemrédi [27] yielded the first exponential bound.

Theorem 2.3. [45] $p(n) \leq .999^n$.

Their arguments were simplified by Tao and Vu in 2004 [44], resulting in a slightly better bound $O(.958^n)$. Shortly afterwards, these authors [45] combined the approach from [27] with the ideas of inverse theorems (coming from additive combinatorics) to obtain the following more significant improvement

Theorem 2.4. [45] $p(n) \leq (3/4 + o(1))^n$.

Very recently, Bourgain et al. [5], improved the bound further to

Theorem 2.5. $p(n) \leq (\frac{1}{\sqrt{2}} + o(1))^n$.

The approach from [45, 5] allows one to deduce bound on p_n from simple trigonometrical estimates. For instance, the 3/4-bound comes from the fact that

$$|\cos x| \leq \frac{3}{4} + \frac{1}{4}\cos 2x,$$

while the $1/\sqrt{2}$ bounds come from

$$|\cos x|^2 = \frac{1}{2} + \frac{1}{2}\cos 2x.$$

The main theorem of [5] ([5, Theorem 2.2]) provides a formal connection between singularity estimates and trigonometrical estimates of this type.

To conclude this section, let us mention a very useful tool, the Littlewood-Offord-Erdös theorem. Let $\mathbf{v} = \{v_1, \dots, v_n\}$ be a set of n non-zero real numbers and $\xi_1, \dots, \xi_n$ be i.i.d. random Bernoulli variables. Define $S := \sum_{i=1}^n \xi_i v_i$ and $p_{\mathbf{V}}(a) = \mathbf{P}(S = a)$ and $p_{\mathbf{V}} = \sup_{a \in \mathbf{Z}} p_{\mathbf{V}}(a)$.

The problem of estimating $p_{\mathbf{V}}$ originated from a paper of Littlewood and Offord in the 1940s [37]. Erdös, improving a result of Littlewood and Offord, proved the following theorem, which we will refer to as the Erdös-Littlewood-Offord inequality.

Theorem 2.6. *Let $v_1, \dots, v_n$ be non-zero numbers and ξ_i be i.i.d. Bernoulli random variables. Then*

$$p_{\mathbf{V}} \leq \frac{\binom{n}{\lfloor n/2 \rfloor}}{2^n} = O(n^{-1/2}).$$

Theorem 2.6 is a classical result in combinatorics and has many non-trivial extensions with far reaching consequences (see [48, Chapter 7] and the references therein).

To give the reader a feeling about how the Littlewood-Offord problem can be useful in estimating p_n, let us consider the following process. We expose the rows of M_n one by one from the top. Assume that the first $n-1$ rows are independent and form a hyperplane with normal vector $\mathbf{v} = (v_1, \dots, v_n)$. Conditioned on these rows, the probability that M_n is singular is

$$\mathbf{P}(X \cdot \mathbf{v} = 0) = \mathbf{P}(\xi_1 v_1 + \cdots + \xi_n v_n = 0),$$

where $X = (\xi_1, \dots, \xi_n)$ is the last row.

In Section 3, the reader will see an application of Theorem 2.6 that leads to Komlós result that $p_n = o(1)$. In order to obtain the stronger estimates in Theorems 2.4 and 2.5, one needs to ebstablish Inverse Littlewood-Offord theorems. These theorems are motivated by inverse theorems of Freiman type in Additive Combinatorics, the discussion of which is beyond the scope of this survey. The interested reader is referred to [48, Chapter 7], [48, Chapter 5] and [45].

3. Simple proof of Komlos' Theorem

Let us start with a simple fact. Here and later Bernoulli vectors are vectors with coordinates ± 1.

Fact 3.1. *Let H be a subspace of dimension $1 \le d \le n$. Then H contains at most 2^d Bernoulli vectors.*

To see this, notice that in a subspace of dimension d, there is a set of d coordinates which determine the others. Let H_i be the space spanned by $X_1, \dots, X_i$. This fact implies

$$\mathbf{P}(singular) \le \sum_{i=1}^{n-1} \mathbf{P}(X_{i+1} \in H_i) \le \sum_{i=1}^{n-1} 2^{i-n} \le 1 - \frac{2}{2^n}.$$

While this bound is the opposite of what we want to prove, we notice that the loss comes at the end. Thus, to obtain the desired upper bound $o(1)$, it suffices to show that the last $\log\log n$ terms are bounded by $\frac{1}{\log^{1/3} n}$. To do this, we will exploit the fact that the H_i are spanned by random vectors. The following lemma (which is a more effective version of the above fact) implies Komlos' Theorem 2.2 via the union bound.

Lemma 3.2. *Let H be the subspace spanned by d random vectors, where $d \ge n - \log\log n$. Then with probability at least $1 - \frac{1}{n}$, H contains at most $\frac{2^n}{\log^{1/3} n}$ Bernoulli vectors.*

We say that a set S of d vectors is *k-universal* if for any set of k different indices $1 \le i_1, \dots, i_k \le n$ and any set of signs $\epsilon_1, \dots, \epsilon_n$ ($\epsilon_i = \pm 1$), there is a vector V in S such that the sign of the i_jth coordinate of V matches ϵ_j, for all $1 \le j \le k$.

Fact 3.3. *If $d \geq n/2$, then with probability at least $1 - \frac{1}{n}$, a set of d random vectors is k-universal, for $k := \log n/10$.*

To prove this, notice that the failure probability is, by the union bound, at most

$$\binom{n}{k}(1 - \frac{1}{2^k})^d \leq n^k(1 - \frac{1}{2^k})^{n/2} \leq n^{-1}.$$

If S is k-universal, then any non-zero vector V in the orthogonal complement of the subspace spanned by S should have more than k non-zero vectors (otherwise, there would be a vector in S having positive inner product with V). If we fix such a V, and let X be a random Bernoulli vector, then by Littlewood-Offord-Erdös Theorem (Theorem 2.6),

$$\mathbf{P}(X \in Span(S)) \leq \mathbf{P}(X \cdot V = 0) = O(\frac{1}{k^{1/2}}) \leq \frac{1}{\log^{1/3} n},$$

proving Lemma 3.2 and Komlos' Theorem 2.2.

4. The singular probability: symmetric case

As an analogue to the previous section, we try to estimate p_n^{sym}, the probability that the symmetric matrix M_n^{sym} is singular.

This problem was mentioned to the first author by G. Kalai and N. Linial (personal conversations) around 2004. To our surprise, at that point, even an analogue of Komlos' 1967 result was not known. The following conjecture, raised by B. Weiss in the 1980s, was open.

Conjecture 4.1. (Weiss' conjecture) $p_n^{sym} = o(1)$.

The main difficulty concerning M_n^{sym} is that its rows are no longer independent. In particular, the last row is almost determined by the previous ones. Thus, the row exposing procedure introduced for the non-symmetric case is no longer useful.

Few years ago, Costello, Tao and Vu [15] found a way to circumvent this problem. It turns out that the right way to build the symmetric matrix M_n^{sym} is not row by row (as for M_n), but corner to corner. Starting with a single entry, one, at each step, adds a random row and its transpose to the existing matrix, increasing its size by one. One is able to prove that with high probability, the co-rank of the resulting matrix, as its size increases, behaves like the end point of a bias random walk on non-negative integers which has a strong tendency to go to the left whenever possible. This leads to a confirmation of Weiss' conjecture.

Theorem 4.2. [15] $p_n^{sym} = O(n^{-1/4})$.

The key technical tool in the proof of Theorem 4.2 is the following (quadratic) variant of Theorem 2.6.

Theorem 4.3. (Quadratic Littlewood-Offord) *Let a_{ij} be non-zero real numbers and ξ_i, $1 \le i, j \le n$ be i.i.d. Bernoulli random variables. Let Q be the quadratic form $Q := \sum_{1\le i,j\le n} a_{ij}\xi_i\xi_j$. Then for any value a*

$$\mathbf{P}(Q = a) = O(n^{-1/4}).$$

With this tool in hand, let us consider the corner-to-corner building process of M_n^{sym}. Assume that the $(n-1)\times(n-1)$ corner M_{n-1}^{sym} has been built. To obtain M_n^{sym}, we add a random row $X = (\xi_1, \dots, \xi_n)$ and its transpose. Conditioning on M_{n-1}^{sym}, the determinant of the resulting $n \times n$ matrix can be expressed as

$$\sum_{1\le i,j\le n-1} a_{ij}\xi_i\xi_j + \det M_{n-1},$$

where a_{ij} (up to the signs) are the cofactors of M_{n-1}. If M_n^{sym} is singular, then its determinant is 0, which implies

$$Q := \sum_{1\le i,j\le n-1} a_{ij}\xi_i\xi_j = -\det M_{n-1},$$

which gives ground for an application of Theorem 4.3.

The bound in Theorem 4.3 is not sharp. Taking $Q = (\xi_1 + \cdots + \xi_n)^2$, we see that (for n even)

$$\mathbf{P}(Q = 0) = \mathbf{P}(\xi_1 + \cdots + \xi_n = 0) = \frac{\binom{n}{n/2}}{2^n} = \Omega(n^{-1/2}).$$

In a recent paper [12], Costello matched this bound. Among others, he showed

Theorem 4.4. *For any fixed $\epsilon > 0$ the following holds for all sufficiently large n. Let a_{ij} be non-zero real numbers and ξ_i, $1 \le i, j \le n$ be i.i.d. Bernoulli random variables. Let Q be the quadratic form $Q := \sum_{1\le i,j\le n} a_{ij}\xi_i\xi_j$. Then for any value a*

$$\mathbf{P}(Q = a) \le n^{-1/2+\epsilon}.$$

As a corollary, Costello [12] improved the upper bound on p_n^{sym} to $n^{-1/2+\epsilon}$. This is the best bound known at this moment. On the other hand, motivated by the non-symmetric case, we conjecture

Conjecture 4.5. $p_n^{sym} = (1/2 + o(1))^n$.

It would already be a major progress to obtain a bound of the form n^{-C}, where C can be set arbitrarily large. Another interesting question is to generalize Costello's quadratic Littlewood-Offord result for higher degree polynomials.

Conjecture 4.6. (Polynomial Littlewood-Offord) *Let d be a fixed positive integer. Consider the polynomial $P = \sum_{1\le i_j \le n} a_{i_1,\dots,i_d}\xi_{i_1}\dots\xi_{i_d}$, where $a_{i_1\dots i_d}, 1 \le i_j \le n$ are non-zero real numbers. Then for any number a*

$$\mathbf{P}(P = a) = O(n^{-1/2}).$$

5. Ranks and co-ranks

The singular probability can be seen as the probability that the random matrix has co-rank at least one. What about a larger co-rank? Let us use $p_{n,k}$ to denote the probability that M_n has co-rank at least k. It is easy to show that

$$p_{n,k} \geq (\frac{1}{2} + o(1))^{kn}. \tag{5.1}$$

It is tempting to conjecture that this bound is sharp for constants k. In [27], Kahn, Komlós and Szemerédi showed

Theorem 5.1. *There is a function $\epsilon(k)$ tending to zero with k such that*

$$p_{n,k} \leq \epsilon^n.$$

An analogue of this is available for the symmetric case (of course only at polynomial level). By refining the arguments in [15], one can show that there is a function $C(k)$ tending to infinity with k such that

$$p_{n,k}^{sym} \leq n^{-C(k)}.$$

In fact, one can take $C(k) = ck$ for some positive constant c. (Thanks to Costello for pointing this out.)

In [5], the authors consider a variant of M_n where the first l rows are fixed and the next $n-l$ are random. Let L be the submatrix defined by the first l rows and denote the model by $M_n(L)$. It is clear that $\mathrm{corank} M_n(L) \geq \mathrm{corank} L$. The authors of [5] showed (see [5, Theorem 1.4] for more details)

Theorem 5.2. *There is a positive constant c such that*

$$\mathbf{P}(\mathrm{corank} M_n(L) > \mathrm{corank} L) \leq (1-c)^n.$$

Let us now go back to the symmetric model M_n^{sym} and view it from a new angle, connecting it to the adjacency of the Erdös-Rényi random graph $G(n, 1/2)$. One can see that

$$M_n^{sym} = 2A(n, 1/2) - J_n,$$

where J_n is the all-one matrix of size n. (Here we allow $G(n, 1/2)$ to have loops, so the diagonal entries of $A(n, 1/2)$ are zero and one. If we fix all diagonal entries to be zero, the analysis does not change essentially.) Since J_n has rank one, it follows from Theorem 4.2 that with probablity $1 - o(1)$, $A(n, 1/2)$ has corank at most one.

One can reduce the co-rank to zero by a slightly trickier argument. Consider M_{n+1}^{sym} instead of M_n^{sym} and normalize so that its first row and column are all-negative one. Adding this matrix with J_{n+1}, we obtain a matrix of the form

$$\begin{pmatrix} 0 & 0 \\ 0 & M_n^{sym} + J_n \end{pmatrix}$$

Thus we conclude

Corollary 5.3. *With probability $1 - o(1)$, $\mathrm{corank} A(n, 1/2) = 0$.*

It is natural to ask if this statement still holds for a smaller density p. The answer is negative after a certain threshold. Indeed, if $p < (1-\epsilon)\log n/n$ for some positive constant ϵ, then $G(n,p)$ has a.s. isolated vertices (see [4, 26]) which means that its adjacency matrix has all zero rows and so is singular. Costello and Vu [13] proved that $\log n/n$ is the right threshold.

Theorem 5.4. [13] *For any constant* $\epsilon > 0$*, with probability* $1 - o(1)$*,*

$$\text{corank} A(n, (1+\epsilon)\log n/n) = 0.$$

For $p < \log n/n$, the co-rank of $A(n,p)$ is no longer zero (with high probability). The behavior of this random variable is not entirely understood. For the case when $p = c\log n/n$ for some constant $0 < c < 1$, Costello and Vu [14] showed that with probability $1 - o(1)$, the co-rank is determined by small subgraphs. Here is an example

Theorem 5.5. [13] *For any constant* $\epsilon > 0$ *and* $(1/2+\epsilon)\log n/n < p < (1-\epsilon)\log n/n$*, with probability* $1-o(1)$*,* $\text{corank} A(n,(1+p) = I(n,p)$*, where* $I(n,p)$ *is the number of isolated vertices in* $G(n,p)$*.*

For the range $p = c/n, c > 1$, $G(n,p)$ consists of a giant component and many small components. It makes sense to focus on the giant one which we denote by $Giant(n,p)$. Since $Giant(n,p)$ has cherries (pair of vertices of degree one with a common neighbor), the adjacency matrix of $Giant(n,p)$ is singular (with high probability). However, if we look at the k-core of $Giant(n,p)$, for $k \geq 3$, it seems plausible that this subgraph has full rank.

Conjecture 5.6. *Let k be a fixed integer at least 3. With probability* $1 - o(1)$*, the adjacency matrix of the k-core of $Giant(n,p)$ is non-singular.*

Let us now consider the random regular graph $G_{n,d}$. For $d = 2$, $G_{n,d}$ is just union of disjoint circles. It is not hard to show that with probability $1 - o(1)$, one of these circles has length divisible by 4, and thus its adjacency matrix is non-singular (in fact, the corank will be $\Theta(n)$ as the number of circles of length divisible by 4 is of this order). Somewhat embarrassingly, we know nothing about the case $d \geq 3$.

Conjecture 5.7. *For any* $3 \leq d \leq n/2$*, with probability* $1-o(1)$ $A_{n,d}$ *is non-singular.*

6. Determinant and permanent

We start with a basic question

What is the determinant of M_n?

Komlós 1967 theorem showed that with probability $1 - o(1)$, M_n has full rank, namely $|\det M_n| > 0$. However, this (and other theorems in Section 2) do not give any non-trivial estimate on $|\det M_n|$.

Notice that all rows of M_n have length $\sqrt{n}$. Hadamard's inequality thus implies that $|\det M_n| \leq n^{n/2}$. It has been conjectured that, with probability close to 1, $|\det M_n|$ is close to this upper bound.

Conjecture 6.1. *Almost surely* $|\det M_n| = n^{(1/2-o(1))n}$.

This conjecture is supported by the following observation of Turán.

Fact 6.2.

$$\mathbf{E}((\det M_n)^2) = n!.$$

To verify this, notice that

$$(\det M_n)^2 = \sum_{\pi,\sigma \in S_n} (-1)^{\text{ sign}\pi + \text{ sign}\sigma} \prod_{i=1}^{n} \xi_{i\pi(i)}\xi_{i\sigma(i)}.$$

By linearity of singularity and the fact that $\mathbf{E}(\xi_i) = 0$, we have

$$\mathbf{E}(\det M_n)^2 = \sum_{\pi \in S_n} 1 = n!.$$

It follows immediately by Markov's bound that for any function $\omega(n)$ tending to infinity with n, almost surely

$$|\det M_n| \leq \omega(n)\sqrt{n!}.$$

In [44], Tao and Vu established the matching lower bound, confirming Conjecture 6.1.

Theorem 6.3. *Almost surely*

$$|\det M_n| \geq \sqrt{n!}\exp(-29\sqrt{n \log n}).$$

We are going to sketch the proof very briefly as it contains a useful lemma. For a more detailed proofs, we refer to [44].

Proof. We view $|\det M_n|$ as the volume of the parallelepiped spanned by n random $\{-1, 1\}$ vectors. This volume is the product of the distances from the $(d+1)$st vector to the subspace spanned by the first d vectors, where d runs from 0 to $n-1$. We are able to obtain a very tight control on this distance (as a random variable), thanks to the following lemma.

Lemma 6.4. *Let W be a fixed subspace of dimension $1 \leq d \leq n-4$ and X a random ± 1 vector. Then*

$$\mathbf{E}(\text{dist}(X, W)^2) = n - d. \tag{6.1}$$

Furthermore, for any $t > 0$

$$\mathbf{P}(|\text{dist}(X, W) - \sqrt{n-d}| \geq t + 1) \leq 4\exp(-t^2/16). \tag{6.2}$$

Observe that in this lemma, we do not need to assume that W is spanned by random vectors. The lemma, however, is not applicable when d is very close to n as it does not imply that the distance is positive almost surely. In this case, we do need to use the assumption that W is random. This assumption allows us to derive information about the normal vector of W, which, combined with Erdös-Littlewood-Offord bound (see Theorem 2.6), provides control on the last few distances. □

Now we turn to the symmetric model M_n^{sym}. Again, by Hadamard's inequality $|\det M_n^{sym}| \le n^{n/2}$. It seems plausible to conjecture

Conjecture 6.5. *With probability* $1 - o(1)$

$$|\det M_n^{sym}| = n^{(1/2 - o(1))n}.$$

Turan's identity no longer holds because of correlation caused by symmetry. However, one can still show

$$\mathbf{E}(\det M_n^{sym})^2 = n^{(1+o(1)n)}.$$

On the other hand, proving a lower bound for $|\det M_n|$ is much more difficult. The above approach (which, similarly to Section 2, expose the matrix row by row) is no longer useful for the symmetric case.

Conjecture 6.5 was confirmed only very recently, as a corollary of the Four Moment Theorem [41]. The basic idea is as follows. Let $0 \le \sigma_1 \le \sigma_2 \le \cdots \le \sigma_n$ be the singular values of M_n^{sym}. Basic facts from linear algebra tell us that

$$|\det M_n^{sym}| = \prod_{i=1}^{n} \sigma_i.$$

The distribution of the σ_i is controlled by Wigner's semi-circle law (see [41, 52]). As a consequence, we have a good estimate for large σ_i (say, $i \ge \epsilon n$ for some small ϵ). The product of these values is basically $n^{(1/2-o(1))n}$. To complete the proof, one would need to control the small singular value. In particular, one needs to bound them from below. This is not non-trivial. (Recall that to even show $\sigma_1 > 0$, which is equivalent to Weiss' conjecture, required some effort). However, the method introduced in [41] was sufficiently powerful to enable us to even compute the limiting distribution of the singular values. This leads to a confirmation of Conjecture 6.5.

Theorem 6.6. *With probability* $1 - o(1)$

$$|\det M_n^{sym}| = n^{(1/2 - o(1))n}.$$

In fact, we can also compute the limiting distribution of $|\det M_n^{sym}|$, properly normalized (see [41] for more details).

Let us now turn to a related notation of permanent. The formal definition of the determinant of a matrix M (with entries $m_{ij}, 1 \leq i, j \leq n$) is

$$\det M := \sum_{\pi \in S_n} (-1)^{\text{sign}\pi} \prod_{i=1}^{n} m_{i\pi(i)}.$$

The permanent of M is defined as

$$\text{Per} M := \sum_{\pi \in S_n} \prod_{i=1}^{n} m_{i\pi(i)}. \tag{6.3}$$

It is easy to see that Turan's identity still holds, namely

$$\mathbf{E}(\text{ Per} M_n)^2 = n!.$$

It suggests that, similar to $|\det M_n|$, $|\text{ Per} M_n|$ is typically $n^{(1/2-o(1))n}$. However, this is much harder to show, and the following conjecture, which can be seen as the permanent variant of Komlós 1967 result, was open for a long time

Conjecture 6.7. *With probablity $1 - o(1)$, the permanent of M_n is non-zero.*

The source of difficulty here is that despite the similarity between the definitions, unlike determinant, permanent does not really have any good geometric of linear algebraic interpretation (which play the key roles in all problems considered so far).

In 2007, Tao and Vu found an entirely combinatorial approach to treat the permanent problem [47]. This approach relies on the formal definition (6.3) and makes heavy use of martingale techniques from probabilistic combinatorics. As a result, they managed to prove an analogue of Theorem 6.3

Theorem 6.8. *With probability $1 - o(1)$*

$$|\ Per M_n| = n^{(1/2-o(1))n}.$$

The missing (final) piece of the picture is (naturally) the symmetric counterpart of Theorem 6.8.

Conjecture 6.9. *With probability $1 - o(1)$*

$$|\ Per M_n^{sym}| = n^{(1/2-o(1))n}.$$

7. Expansion and the second eigenvalue

Let G be a graph on n points and A its adjacency matrix. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of A. If G is d-regular, then $\lambda_1 = d$. In this case, a critical parameter of the graph is

$$\lambda(G) := \max\{|\lambda_2|, |\lambda_n|\}.$$

One can derive many interesting properties of the graph from the value of this parameter. The general phenomenon here is that if $\lambda(G)$ is significantly less than

d, then the edges of G distribute like in a random graph with edge density d/n [2, 8]. A representative fact is the following [3]. Let A, B be sets of vertices and $E(A, B)$ the number of edges with one end point in A and the other in B, then

$$|E(A,B) - \frac{d}{n}|A||B|| \leq \lambda(G)\sqrt{|A||B|}. \tag{7.1}$$

Notice that the term $\frac{d}{n}|A||B|$ is the expectation of the number of edges between A and B if G is random (in the Erdös-Rényi sense) with edge density d/n. Graphs with small λ are often called *pseudo-random* [8, 31].

One can use this information about edge distribution to derive various properties of the graphs (see [31] for many results of this kind). The whole concept can be generalized for non-singular graphs. In this case, one needs to consider the Laplacian rather than the adjacency matrix (see, for example, [9]).

From (7.1), it is clear that the smaller λ is the more similar G is to a random graph. But how small can λ be?

It was proved by Alon and Boppana [1], that if d is fixed and n tends to infinity, then

$$\lambda(G) \geq 2\sqrt{d-1} - o(1).$$

Graphs which satisfy $\lambda(G) < 2\sqrt{d-1}$ are called Ramanujan graphs. It is very hard to construct such graphs, and all known constructions, such as those by Lubotzky-Phillip-Sarnak [33] and Margulis [34] rely heavily on number theoretic results.

On the other hand, through numerical experiments, we have reason to believe that random regular graphs are Ramanujan with a decent probability.

Alon [1] conjectured that for any fixed d, a.s.

$$\lambda_2(G_{n,d}) = 2\sqrt{d-1} + o(1).$$

Friedman [22] and Kahn and Szemerédi [**?**] showed that if d is fixed and n tends to infinity, then a.s. $\lambda(G_{n,d}) = O(\sqrt{d})$. Recently, Friedman, in a highly technical paper [23], proved Alon conjecture. In fact, he proved the stronger statement that a.s. $\lambda(G_{n,d}) = 2\sqrt{d-1} + o(1)$.

Theorem 7.1. [23] *(Second eigenvalue of random regular graphs with fixed degree) For any fixed d and n tending to infinity, a.s.*

$$\lambda(G_{n,d}) = (2 + o(1))\sqrt{d-1}.$$

What happens if d also tends to infinity with n? It is not clear (at least to us) that the proof in [23] can be extended to this case. On the other hand, it is not hard to show that $\lambda(G(n,p))$, where $G(n,p)$ is the Erdös-Réyi random graph, is $(2 + o(1))\sqrt{np(1-p)}$ for sufficiently large p (e.g., $p \geq n^{-1+\epsilon}$ for any fixed $0 < \epsilon < 1$). Motivated by the universality principle, we make the following conjecture

Conjecture 7.2. *Assume that $d \le n/2$ and both d and n tend to infinity. Then a.s.*

$$\lambda(G_{n,d}) = (2 + o(1))\sqrt{d(1 - d/n)}.$$

Nilli [38] showed that for any d-regular graph G having two edges with distance at least $2k + 2$ between them $\lambda_2(G) \ge 2\sqrt{d-1} - 2\sqrt{d-1}/(k+1)$. If $d = n^{o(1)}$ then $G_{n,d}$ has diameter $\omega(1)$ with probability one. Thus in this case

$$\lambda(G_{n,d}) \ge \lambda_2(G_{n,d}) \ge (2 + o(1))\sqrt{d}$$

with probability one. This proves the lower bound in Conjecture 7.2. For a general d, it is easy to show (by computing the trace of the square of the adjacency matrix) that any d-regular graph G on n vertices satisfies

$$\lambda(G) \ge \sqrt{d(n-d)/(n-1)} \approx \sqrt{d(1 - d/n)}.$$

(We would like to thank N. Alon for pointing out this bound.)

Let us now turn to the upper bound. For $d = o(n^{1/2})$, one can follow Kahn-Szemerédi approach to show that $\lambda(G_{n,d}) = O(\sqrt{d})$ a.s. For larger d, there is a weaker bound $o(d)$ [32, Theorem 2.8] proved by the trace method. The following two approaches look promising

- (Suggested by Krivelevich) Combine the sharp concentration result in the previous section with the probability that a random graph is regular. Using this, one can show for example that $\lambda(G_{n,d}) = O(\sqrt{d \log n})$ for d close to n ($d = n/2$, for instance).
- The Sandwich Theorem [28] (which basically states that $G_{n,d}$ has the same behavior as $G(n,p)$ under some conditions) implies

  $$\lambda(G_{n,d}) = \lambda(G(n, d/n)) + O(\sqrt{d \log n}).$$

  For most values of d, $\lambda(G(n, d/n)) = O(\sqrt{d})$. Thus, if the Sandwich conjecture [28] holds, it would imply upper bound $O(\sqrt{d \log n})$ for most values of d.

We feel confident that we can prove that $\lambda(G_{n,d}) = O(\sqrt{d \log n})$ for all d using these approaches. However, removing the log term seems tricky and (in our opinion) Conjecture 7.2 may be hard. In fact, even the following special and weakened case looks already challenging

Problem 7.3. *Prove that $\lambda(G_{n,n/2}) = O(\sqrt{d})$.*

8. Properties of eigenvectors

If M is symmetric, then its unit eigenvectors form an orthonormal basis. In [41, 42], it is shown that

Theorem 8.1. *With probability $1 - o(1)$,*

$$\max \|v\|_\infty \le n^{-1/2} \log^{20} n,$$

where the maximum is taken over all unit eigenvectors of M_n^{sym}.

Since any unit vector v satisfies $\|v\|_\infty \geq n^{-1/2}$, this bound is best possible up to the logarithmic term. The same result holds for the non-symmetric model M_n, with respect to singular vectors instead of eigenvectors [43].

The situation with the adjacency matrix of a random graph is somewhat more complicated. Consider $A(n,p)$ with $p = \Theta(1)$. The sum of any rows is close to np. It suggests that the largest eigenvalue λ_1 of $A(n,p)$ is approximately np and its corresponding eigenvector v_1 is close to $\frac{1}{\sqrt{n}}v_0$, where v_0 is the all-one vector. This intuition was confirmed by Komlós and Füredi [24]. Recently Mitra [36] improved the entry-wise bound on the first eigenvector of $A(n,p)$.

Theorem 8.2. *Assume $p \geq \log^6 n/n$, then for all $i \in [n]$ there is a constant c such that*

$$\left|v_1(i) - \frac{1}{\sqrt{n}}\right| \leq c\frac{1}{\sqrt{n}}\frac{\log n}{\log(np)}\sqrt{\frac{\log n}{np}}$$

with probability $1 - o(1)$.

In [16], Dekel, Lee and Linial, motivated by the study of nodal domains, raised the following question.

Question 8.3. *Is it true that almost surely every eigenvector u of $G(n,p)$ has $||u||_\infty = n^{-1/2+o(1)}$?*

The bound $n^{-1/2+o(1)}$ was also conjectured by the second author of this paper in an NSF proposal (submitted Oct 2008). He and Tao [41] proved this bound for eigenvectors corresponding to the eigenvalues in the bulk of the spectrum for the case $p = 1/2$. If one defines the adjacency matrix by writing -1 for non-edges, then this bound holds for all eigenvectors [41, 42].

The above two questions were raised under the assumption that p is a constant in the interval $(0,1)$. For p depending on n, the statements may fail. If $p \leq \frac{(1-\epsilon)\log n}{n}$, then the graph has (with high probability) isolated vertices and so one cannot expect that $\|u\|_\infty = o(1)$ for every eigenvector u.

Numerical evidence suggests the conjecture to be true. Additionally, we can make an even stronger conjecture

Conjecture 8.4. *Assume $p \geq \frac{(1+\epsilon)\log n}{n}$ for some constant $\epsilon > 0$. Let v be a random unit vector whose distribution is uniform in the n-dimensional unit square. Let u be a unit eigenvector (not corresponding to the largest eigenvector) of $G(n,p)$ and w be a constant n-dimensional vector. Then for any $\delta > 0$*

$$\mathbf{P}(|w \cdot u - w \cdot v| > \delta) = o(1).$$

The proof of Theorem 8.1 does not apply directly to the model $A(n,p)$. If p is a constant, then it is shown in [41] that almost surely all eigenvectors corresponding to eigenvalues in the bulk of the spectrum do satisfy the above bound.

The situation with sparser random graphs is less clear. Komlós and Füredi proved that all other eigenvalues of $A(n,p)$ have absolute value at most $(2 +$

$o(1))\sqrt{np}$ (almost surely). Furthermore, it is not hard to show that they distributed according to Wigner's semi-circle law. All these results extend to the case $p \geq n^{-1+\epsilon}$ with no difficulty.

By modifying the arguments from [41, 42], Tran, Vu and Wang [49] proved that

Theorem 8.5. *[(Infinity norm of eigenvectors)] For $p = \omega(\log n/n)$, there exists an orthonormal basis of eigenvectors of $A(n,p)$, $\{u_1, \dots, u_n\}$, such that for every $1 \leq i \leq n$, $||u_i||_\infty = o(1)$ almost surely.*

For Questions 8.3, they obtain a good quantitative bound for those eigenvectors which correspond to eigenvalues bounded away from the edge of the spectrum.

Theorem 8.6. *Assume $p = g(n)\log n/n \in (0,1)$, where $g(n)$ can tend to ∞ arbitrarily slowly. Let $B_n = \frac{1}{\sqrt{n}\sigma}A_n$. For any $\kappa > 0$, and any $1 \leq i \leq n$ with $\lambda_i(B_n) \in [-2+\kappa, 2-\kappa]$, there exists a corresponding eigenvector u_i such that $||u_i||_\infty = O_\kappa(\sqrt{\frac{\log g(n)^{2.2} \log n}{np}})$with overwhelming probability.*

Let us now consider random regular graphs. Recently Dimitriu and Pal [17] proved the following result. Let $d = \log^\gamma n$ for a constant $0 < \gamma < 1$, and set $\eta_n := \frac{6(\log d)^{1+\sigma}}{\sqrt{\log n}}$ where $\sigma > 0$ is a constant. A unit vector $v = (v_1, \dots, v_n)$ is (T, ϵ)-localized if there is a set X of size T such that $\sum_{i \in X} v_i^2 \geq \epsilon$.

Theorem 8.7. *For any fixed $\epsilon > 0$, almost surely, no eigenvector of $A(n,d)$ is $(o(\eta_n^{-1}), \epsilon)$-localized.*

A more recent result of Brooks and Lindenstrauss [6] showed

Theorem 8.8. *Let d, ϵ be constants. Then there is a constant $\delta = \delta(d, \varepsilon) > 0$ such that the following holds. Almost surely, no eigenvector of $A(n,d)$ is (n^δ, ϵ) localized.*

In fact, Brooks and Lindenstrauss result holds for deterministic graphs, under a condition on short cycles, which hold almost surely for regular random graphs with constant degree.

Problem 8.9. *Can we replace the (n^δ, ϵ) -localization in Theorem 8.8 by $(\delta n, \epsilon)$-localization?*

9. Random regular graphs

The random d-regular graph $G_d(n)$ is obtained by taking a graph uniformly at random from the set of all simple d-regular graphs on n vertices. While this definition looks simple, it, unfortunately, does not posses the powerful features of the previous one. In particular, there is no obvious relation to probability theory as in the Erdös-Rényi model. Consequently, compared to the study of Erdös-Rényi model, the study of random regular graphs relies on different techniques, usually of enumerative nature.

We now discuss the spectral properties of the adjacency matrix of regular random graph. In 1950s, Wigner [52] discovered the famous semi-circle for the limiting distribution of the eigenvalues of random matrices. His proof extends, without difficulty, to the adjacency matrix of $G(n, p)$, given that $np \to \infty$ with n. (See Figure 1 for a numerical simulation).

Theorem 9.1. *For $p = \omega(\frac{1}{n})$, the empirical spectral distribution (ESD) of the matrix $\frac{1}{\sqrt{n}\sigma} A_n$ converges in distribution to the semicircle law which has a density $\rho_{sc}(x)$ with support on $[-2, 2]$,*

$$\rho_{sc}(x) := \frac{1}{2\pi}\sqrt{4 - x^2}.$$

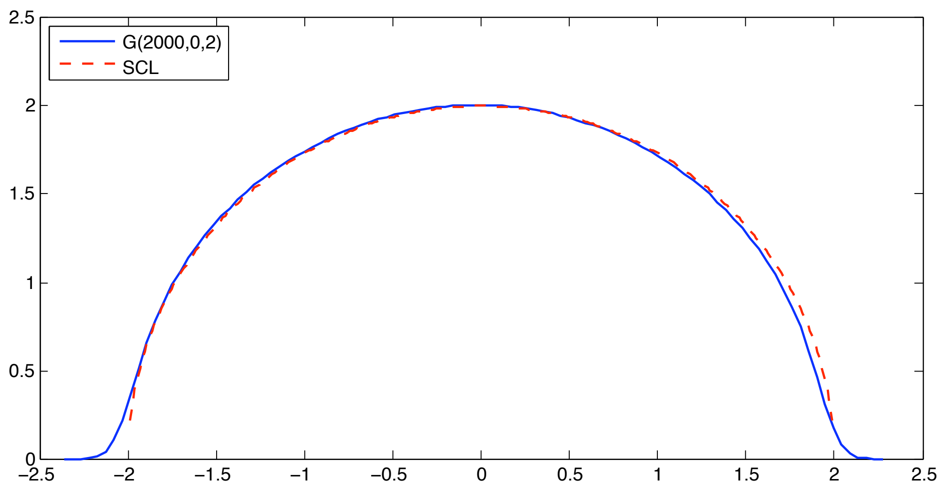

FIGURE 1. The probability density function of the ESD of $G(2000, 0.2)$

If $np = O(1)$, the semicircle law no longer holds. In this case, the graph almost surely has $\Theta(n)$ isolated vertices, so in the limiting distribution, the point 0 will have positive constant mass.

The case of random regular graph, $G_{n,d}$, was considered by McKay [35] about 30 years ago. He proved that if d is fixed, and $n \to \infty$, then the limiting density function is

$$f_d(x) = \begin{cases} \frac{d\sqrt{4(d-1)-x^2}}{2\pi(d^2-x^2)}, & \text{if } |x| \le 2\sqrt{d-1}; \\ 0 & \text{otherwise.} \end{cases}$$

This is usually referred to as McKay or Kesten-McKay law. It is easy to verify that as $d \to \infty$, if we normalize the variable x by $\sqrt{d-1}$, the above density converges to the semicircle law on $[-2, 2]$. It is thus natural to conjecture that Theorem 9.1 holds for $G_{n,d}$ with $d \to \infty$. Let A'_n be the adjacency matrix of $G_{n,d}$., and the normalized version

$$M'_n = \frac{1}{\sqrt{d}}(A'_n - \frac{d}{n}J).$$

Conjecture 9.2. *If $d \to \infty$ then the ESD of $\frac{1}{\sqrt{n}}M'_n$ converges to the standard semicircle law.*

Dimitriu and Pal [17] showed that the conjecture holds for d tending to infinity very slowly, $d = n^{o(1)}$. Their proof which used trace method does not seem to work for larger d because it depends on the "local tree-like" property, which states that with high probability, most vertices in a random regular graph will have an increasing neighborhood which is free of any cycles. But when d is large ($d \sim n^c$) the graph $G_{n,d}$ will have many short cycles and the local tree-like property will fail.

Very recently, Tran, Vu and Wang [49] proved Conjecture 9.2 in full generality, using a completely different method.

Theorem 9.3. *If d tends to infinity as n goes to infinity, then the empirical spectral distribution of $\frac{1}{\sqrt{n}}M'_n$ converges in distribution to the semicircle distribution.*

The idea of the proof is in order to show that an event in $G_{n,d}$ has probability $o(1)$, one needs to compare the probability of the same event in $G(n, p)$ with the probability that $G(n, p)$ is $np-$regular. If the latter is much larger than the former then we are done. This method allows one to take advantage of various probabilistic tools developed for $G(n, p)$ to prove some results in $G_{n,d}$.

10. Miscellany

About 10 years ago, Krivelevich asked us the following question: Is it true that (with probability $1 - o(1)$), $A(n, 1/2)$ does not have any multiple eigenvalues?

We do not know how to settle this problem, but strongly believe that the answer is affirmative, and the same must hold for other models of random matrices.

Conjecture 10.1. *With probability $1 - o(1)$,*

- *$A(n, 1/2)$ does not have multiple eigenvalues.*
- *M_n does not have multiple eigenvalues.*
- *M_n does not have multiple singular values.*
- *M_n^{sym} does not have multiple eigenvalues.*
- *M_n^{sym} does not have multiple singular values.*

Conjecture 10.2. *With probability $1-o(1)$, the characteristic polynomial of M_n is irreducible.*

With P. Wood, we came up with this conjecture few years ago. Recently, L. Babai informed us he made the same conjecture (unpublished) in the 1970s.

Given $\{-1, 1\}$ matrix M, we denote by $Res(M)$ the minimum number of entries we need to switch (from 1 to -1 and vice versa) in order to make M singular. If M is a sample of M_n, it is easy to show that $Res(M)$ is, a.s. at most $(1/2+o(1))n$, as we can, a.s. change that many entries in the first row to make the first two rows equal. We conjecture that this is the best one can do.

Conjecture 10.3. *Almost surely* $Res(M_n) = (1/2 + o(1))n$*?*

A closely related question (motivated by the notion of local resilience from [40]) is the following. Call a $\{-1, 1\}$ (n by n) matrix M *good* if all matrices obtained by switching (from 1 to -1 and vice versa) the diagonal entries of M are non-singular (there are 2^n such matrices).

Conjecture 10.4. *Almost surely* M_n *is good.*

References

[1] N. Alon, Eigenvalues and expanders, *Combinatorica* **6** (2) (1986), 83–96.

[2] N. Alon and V. Milman, λ_1- isoperimetric inequalities for graphs, and supercon- centrators, *J. Combin. Theory* Ser. B **38** (1) (1985), 73–88.

[3] N. Alon and J. Spencer, *The Probabilistic Method*, 3rd ed., John Wiley & Sons Inc., Hoboken, NJ, 2008.

[4] B. Bollobás, *Random Graphs*, Second edition, Cambridge Studies in Advanced Mathematics, 73. Cambridge University Press, Cambridge, 2001.

[5] J. Bourgain, V. Vu and P. M. Wood, On the singularity probability of discrete random matrices, *J. Funct. Anal.* **258** (2) (2010), 559–603.

[6] S. Brooks and E. Lindenstrauss, Non-localization of eigenfunctions on large regular graphs, arXiv 912.

[7] F. Chung, L. Lu and V. Vu, The spectra of random graphs with expected degrees, *Proc. Nat. Acad. Sci. U.S.A.* **100** (11) (2003), 6313–6318.

[8] F. R. K. Chung, R. L. Graham and R. M. Wilson, Quasi-random graphs, *Combinatorica* **9** (4) (1989), 345–362.

[9] F. Chung, Spectral graph theory, *CBMS series*, no. 92, 1997.

[10] C. Cooper, A. Frieze, M. Molloy and B. Reed, Perfect matchings in random r-regular, s-uniform hypergraphs, *Combin. Probab. Comput.* **5** (1996), 1–14.

[11] C. Cooper, A. Frieze and B. Reed, Random regular graphs of non-constant degree: connectivity and Hamiltonicity, *Combin. Probab. Comput.* **11** (3) (2002), 249–261.

[12] K. Costello, Bilinear and quadratic variants on the Littlewood-Offord problem, *submitted.*

[13] K. Costello and V. Vu, The ranks of random graphs, *Random Structures and Algorithm.* **33** (2008), 269–285.

[14] K. Costello and V. Vu, The rank of sparse random matrices, *submitted.*

[15] K. Costello, T. Tao and V. Vu, Random symmetric matrices are alsmot surely singular, *to appear in Duke Math. Journal.*

[16] Y. Dekel, J. Lee and N. Linial, Eigenvectors of random graphs: Nodal domains, *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques* **4627** (2007), 436–448.

[17] I. Dimitriu and S. Pal, Sparse regular random graphs: spectral density and eigenvectors, *submitted.*

[18] A. Edelman, Eigenvalues and condition numbers of random matrices, *SIAM J. Matrix Anal. Appl.* **9** (4) (1988), 543–560.

[19] A. Edelman and B. Sutton, Tails of condition number distributions, *submitted.*

[20] P. Erdös, On a lemma of Littlewood and Offord, *Bull. Amer. Math. Soc.* **51** (1945), 898–902.

[21] P. Erdös, Extremal problems in number theory, *Proc. Sympos. Pure Math.* **VIII** (1965), 181–189. Amer. Math. Soc., Providence, R.I.
[22] J. Friedman, On the second eigenvalue and random walks in random d-regular graphs, *Technical Report* CX-TR-172-88, Princeton University, August 1988.
[23] J. Fiedman, A proof of Alon's second eigenvalue conjecture and related problems (English summary), *Mem. Amer. Math. Soc.* **195** (910) (2008), viii+100 pp.
[24] Z. Füredi and J. Komlós, The eigenvalues of random symmetric matrices,*Combinatorica* **1** (3) (1981), no. 3, 233–241.
[25] G. Halász, Estimates for the concentration function of combinatorial number theory and probability, *Period. Math. Hungar.* **8** (3-4) (1977), 197–211.
[26] S. Janson, T. Luczak and A. Rucinski, *Random Graphs*, Wiley-Interscience, 2000.
[27] J. Kahn and J. Komlós, E. Szemerédi, On the probability that a random ± 1 matrix is singular, *J. Amer. Math. Soc.* **8** (1995), 223–240.
[28] J. H. Kim and V. Vu, Sandwiching random graphs: Universality between random models, *Advances in Mathematics* **188** (2004), 444-469.
[29] J. H. Kim, B. Sudakov and V. H. Vu, One the asymmetry of random graphs and random regular graphs,*Random Structures and Algorithms* **21** (2002), 216–224.
[30] J. Komlós, On the determinant of $(0, 1)$ matrices, *Studia Sci. Math. Hungar.* **2** (1967), 7–22.
[31] M. Krivelevich and B. Sudakov, Pseudo-random graphs, *More sets, graphs and numbers*, 199-262, Bolyai Soc. Math. Stud., 15, Springer, Berlin, 2006.
[32] M. Krivelevich, B. Sudakov, V. Vu and N. Wormald, Random regular graphs of high degree,*Random Structures and Algorithms* **18** (2001), 346–363.
[33] A. Lubotzky, R. Phillips and P. Sarnak, Ramanujan graphs, *Combinatorica* **8** (3) (1988), 261–277.
[34] G. A. Margulis, Explicit group-theoretical constructions of combinatorial schemes and their application to the design of expanders and superconcentrators, *Problemy Peredachi Informatsii* **24** (1988), 51–60.
[35] B. D. McKay, The expected eigenvalue distribution of a large regular graph, *Linear Algebra and its Applications,* **40** (1981), 203–216.
[36] P. Mitra, Entrywise bounds for eigenvectors of random graphs, *Electron. J. Combin.* **16** (1) (2009), Research Paper 131.
[37] J. E. Littlewood and A. C. Offord, On the number of real roots of a random algebraic equation. III. *Rec. Math. [Mat. Sbornik] N.S.* **12** (1943), 277–286.
[38] A. Nilli, On the second eigenvalue of a graph, *Discrete Mathematics* **91** (1991), 207–210.
[39] A. Sárközy and E. Szemerédi, Uber ein Problem von Erdös und Moser, *Acta Arithmetica* bf 11 (1965), 205–208.
[40] B. Sudakov and V. Vu, Resilience of graphs, *submitted.*
[41] T. Tao and V. Vu, Random matrices: Universality of the local eigenvalues statistics pdf file, *(to appear in Acta Mathematica).*
[42] T. Tao and V. Vu, Random matrices: Universality of local eigenvalue statistics up to the edge, *to appear in Communications in Mathematical Physics.*
[43] T. Tao and V. Vu, Random matrices: The distribution of the smallest singular values (Universality at the hard Edge), *to appear in GAFA.*
[44] T. Tao and V. Vu, On random ± 1 matrices: Singularity Determinant, *to appear in Random Structures and Algorithms.*
[45] T. Tao and V. Vu, On the singularity probability of random Bernoulli matrices, *to appear in J. Amer. Math. Soc..*
[46] T. Tao and V. Vu, Inverse Littlewood-Offord theorems and the condition number of random matrices, *Annals of Math.* **169** (2009), 595–632.
[47] T. Tao and V. Vu, On the permanent of random Bernoulli matrices,*Advances in Mathematics* **220** (2009), 657–669.
[48] T. Tao and V. Vu, *Additive Combinatorics*, Cambridge Univ. Press, 2006.
[49] L. Tran, V. Vu and K. Wang, Sparse random graphs: Eigenvalues and Eigenvectors, *submitted.*

[50] V. Vu, Spectral norm of random matrices, *Combinatorica* **27** (6) (2007), 721–736.
[51] V. Vu, Random discrete matrices, *Horizons of Combinatorics*, Bolyai Society Mathematical Studies **17** (2008), 257–280.
[52] E. P. Wigner. On the distribution of the roots of certain symmetric matrices, *Annals of Mathematics,* **67** (2) (1958), 325–327.
[53] N. C. Wormald, Models of random regular graphs, In: *Surveys in Combinatorics,* J. D. Lamb and D. A. Preece, eds, 1999, pp. 239–298.

Department of Mathematics, Rutgers
New Jersey, NJ 08854, America

E-mail address: linhtran@math.rutgers.edu

E-mail address: vanvu@math.rutgers.edu