ACTA MATHEMATICA VIETNAMICA
Volume 35, Number 2, 2010, pp. 303–317

ON THE GENERALIZED CONVOLUTION FOR THE FOURIER SINE AND THE KONTOROVICH-LEBEDEV TRANSFORMS

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Abstract. A generalized convolution for the Fourier sine transform and the Kontorovich - Lebedev integral transform is introduced. Several properties of this generalized convolution and its application to solve certain systems of integral equations are considered.

1. Introduction

Convolutions and generalized convolutions for various integral transforms received great attention from mathematicians in the last years. For recent surveys and related works on the subject we refer the reader to $[4-10, 12, 13]$. In this paper we introduce a generalized convolution for the Fourier sine transform and the Kontorovich-Lebelev integral transform (see (3.1)). We study several properties of this new generalized convolution and apply them to solve some systems of integral equations in closed form.

This paper is organized as follows. In Section 3 we introduce some function spaces in which the introduced convolution (3.1) is meaningful, and then we prove the factorization property for this convolution (3.2). Some relations of the generalized convolution (3.1) with well-known ones are also proved. Algebraic properties of this new convolution are demonstrated. For this purpose, in Section 2 we recall some well-known convolutions and their properties. In Section 4 we apply the generalized convolution (3.1) to solve in closed form two systems of integral equations, which seem to be difficult to solve by other techniques.

2. Well-known Convolutions

Let $f, g \in L_1(R)$, the convolution for the Fourier integral transform is defined by (see [11])

$$(f \underset{F}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(y) f(x-y) dy, \quad x \in R. \tag{2.1}$$

Received August 12, 2008; in revised form September 9, 2009.
1991 *Mathematics Subject Classification.* 44A35, 44A38, 45E10.
Key words and phrases. Polyconvolution, Fourier sine, Fourier cosine, Kontorovich-Lebedev transform, Toeplitz plus Hankel integral equations, system of integral equations.
This research is partially supported by the Vietnam National Foundation for Science and Technology Development.

We have

$$F(f \underset{F}{*} g)(y) = (Ff)(y)(Fg)(y), \quad \forall y \in R. \tag{2.2}$$

Here F is the Fourier transform (see [3, 11])

$$(Ff)(y) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} f(x) e^{-ixy} dx, \ x \in R.$$

For $f, g \in L_1(R_+)$, we define the generalized convolution for the Fourier sine and cosine transforms by [11]

$$(f \underset{1}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{0}^{+\infty} f(u)\big(g(|x-u|) - g(x+u)\big) du, \quad x > 0, \tag{2.3}$$

for which we have the factorization formula

$$F_s(f \underset{1}{*} g)(y) = (F_s f)(y)(F_c g)(y), \quad \forall y > 0. \tag{2.4}$$

Here F_c, F_s respectively denote the Fourier cosine and the Fourier sine transforms (see [3, 11])

$$(F_c f)(y) = \sqrt{\frac{2}{\pi}} \int\limits_{0}^{+\infty} f(x) \cos(xy) dx, \ \ (F_s f)(y) = \sqrt{\frac{2}{\pi}} \int\limits_{0}^{+\infty} f(x) \sin(xy) dx, \ \ y > 0.$$

We note that the generalized convolution for the Fourier sine and the Fourier cosine transforms was introduced in 1951 by I.N. Sneddon [11]. Further, the convolution of two functions $f, g \in L_1(R_+)$ with a weight function $\eta(y) = \sin y$ for the Fourier sine transform is defined by [4,6]

$$\begin{aligned} \big(f \overset{\eta}{\underset{F_s}{*}} g\big)(x) = \frac{1}{2\sqrt{2\pi}} \int\limits_{0}^{+\infty} & f(x)\big[g(x+t+1) + g(|x-t+1|)\operatorname{sign}(x-t+1) \\ & + g(|x+t-1|)\operatorname{sign}(x+t-1) + g(|x-t-1|)\operatorname{sign}(x-t-1)\big] dt, \ \ x > 0, \end{aligned} \tag{2.5}$$

for which we have

$$F_s\big(f \overset{\eta}{\underset{F_s}{*}} g\big)(y) = \eta(y)(F_s y)(y)(F_s g)(y), \quad \forall y > 0. \tag{2.6}$$

The convolution for the Kontorovich-Lebedev transform of $f, g \in L_p(R_+)$ was studied in [4, 14, 15]

$$(f \underset{K}{*} g)(x) = \frac{1}{2x} \int\limits_{0}^{+\infty} \int\limits_{0}^{+\infty} \exp\Big[-\frac{1}{2}\Big(\frac{xu}{v} + \frac{xv}{u} + \frac{uv}{x}\Big)\Big] f(u) g(v) du dv, \ x > 0, \tag{2.7}$$

for which the following factorization equality holds

$$K(f \underset{K}{*} g)(y) = (Kf)(y)(Kg)(y), \quad \forall y > 0, \tag{2.8}$$

here K is the modified Kontorovich-Lebedev transform [3, 15]

$$(Kf)(y) = \frac{2}{\pi^2} \int_0^{+\infty} K_{iy}(x) x^{-1} f(x) dx, \ y > 0, \tag{2.9}$$

with $K_{iy}(x)$ being the Macdonald function (formula 1.98 [15], p. 14)

$$K_{iy}(x) = \int_0^{+\infty} e^{-xcoshu} \cos yu \, du, \ y \geqslant 0, \ x > 0.$$

The generalized convolution for the Fourier cosine and the Fourier sine transforms for $f, g \in L_1(R_+)$ is introduced in [7]

$$(f \underset{2}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(u) \big[\operatorname{sign}(u-x) g(|u-x|) + g(u+x) \big] du, \quad x > 0, \tag{2.10}$$

which satisfies the factorization identity

$$F_c(f \underset{2}{*} g)(y) = (F_s f)(y)(F_s g)(y), \quad \forall y > 0. \tag{2.11}$$

In [9] we introduce the following two generalized convolutions with the weight function $\gamma_1(y) = y$ for the Kontorovich-Lebedev, Fourier sine and the Fourier cosine transforms for $f \in L_1\Big(R_+, \frac{1}{x}\Big)$ and $g \in L_1\big(R_+\big)$

$$\begin{aligned} \big(f \overset{\gamma_1}{\underset{3}{*}} g\big)(x) = \frac{1}{\pi^2} \int_0^{+\infty} \int_0^{+\infty} & \big[sinh(x+v) e^{-ucosh(x+v)} + \\ & + sinh(x-v) e^{-ucosh(x-v)} \big] f(u) g(v) du dv, \quad x > 0, \end{aligned} \tag{2.12}$$

$$\begin{aligned} \big(f \overset{\gamma_1}{\underset{4}{*}} g\big)(x) = \frac{1}{\pi^2} \int_0^{+\infty} \int_0^{+\infty} & \big[sinh(x+v) e^{-ucosh(x+v)} - \\ & - sinh(x-v) e^{-ucosh(x-v)} \big] f(u) g(v) du dv, \quad x > 0, \end{aligned} \tag{2.13}$$

which respectively satisfy the factorization identities:

$$F_s(f \overset{\gamma_1}{\underset{3}{*}} g)(y) = \gamma_1(y)(Kf)(y)(F_c g)(y), \quad \forall y > 0, \tag{2.14}$$

$$F_c(f \overset{\gamma_1}{\underset{4}{*}} g)(y) = \gamma_1(y)(Kf)(y)(F_s g)(y), \quad \forall y > 0. \tag{2.15}$$

Here K denotes the Kontorovich-Lebedev transform (2.9).

3. A new Generalized Convolution for the Fourier sine Transform and the Kontorovich-Lebedev Transform

Denote by $L_1\big(R_+, \frac{1}{\sqrt{x^3}}\big)$ and $L_1(R_+)$ respectively the set of all functions f and g defined on $(0, +\infty)$ such that

$$\int_0^{+\infty} \frac{1}{\sqrt{x^3}}|f(x)|dx < +\infty \quad \text{and} \quad \int_0^{+\infty} |g(x)|dx < +\infty.$$

Definition 1. *The generalized convolution of two functions f and g for the Fourier sine transform and the Kontorovich-Lebedev integral transform is defined as follows*

(3.1)
$$(f * g)(x) = \frac{1}{\pi^2} \int_0^{+\infty} \int_0^{+\infty} \frac{1}{u}\big[e^{-ucosh(x-v)} - e^{-ucosh(x+v)}\big] f(u)g(v)dudv, \quad x > 0.$$

The following theorem gives a quite different result from [14] where in the factorization equality only one integral transform is involved.

Theorem 3.1. *Suppose that $f \in L_1\Big(R_+, \dfrac{1}{\sqrt{x^3}}\Big)$ and $g \in L_1\big(R_+\big)$. Then the generalized convolution $(f{*}g)(x)$ belongs to $L_1(R_+)$ and satisfies the following factorization property*

$$F_s(f{*}g)(y) = (Kf)(y)(F_s g)(y), \quad \forall y > 0. \tag{3.2}$$

Here, K denotes the Kontorovich - Lebedev transform (9).

Proof. Since $coshx \geqslant 1$, we have $e^{-ucoshx} \leqslant e^{-u}$, for any $x \in R$ and any $u > 0$. Hence

$$\Big|\int_0^{+\infty} \int_0^{+\infty} \frac{1}{u}\big[e^{-ucosh(x-v)} - e^{-ucosh(x+v)}\big] f(u)g(v)dudv\Big|$$
$$\leqslant \int_0^{+\infty} \int_0^{+\infty} \frac{1}{u}e^{-ucosh(x-v)}\, |f(u)|\, |g(v)|dudv + \int_0^{+\infty} \int_0^{+\infty} \frac{1}{u}e^{-ucosh(x+v)}\, |f(u)|\, |g(v)|dudv$$
$$\leqslant 2\int_0^{+\infty} \int_0^{+\infty} \frac{1}{u}e^{-u}\, |f(u)|\, |g(v)|dudv.$$

Furthermore, by hypothesis $f(u) \in L_1\Big(R_+, \frac{1}{\sqrt{u^3}}\Big)$, it follows that $f(u) \in L_1\Big(R_+, \frac{e^{-u}}{u}\Big)$. Therefore,

$$\begin{aligned}\Big|\big(f*g\big)(x)\Big| =& \frac{1}{\pi^2}\Big|\int_0^{+\infty}\int_0^{+\infty}\frac{1}{u}\big[e^{-ucosh(x-v)} - e^{-ucosh(x+v)}\big]f(u)g(v)dudv\Big| \\ \leqslant& \frac{2}{\pi^2}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{u}e^{-u}\,|f(u)|\,|g(v)|dudv \\ =& \frac{2}{\pi^2}\int_0^{+\infty}\frac{e^{-u}}{u}\,|f(u)|\,du\int_0^{+\infty}|g(v)|\,dv < +\infty. \qquad (3.3)\end{aligned}$$

This shows the existence of the generalized convolution (3.1).

Again, since $cosh(x-v) \geqslant \frac{(x-v)^2}{2}$, we have $e^{-ucosh(x-v)} \leqslant e^{-u\frac{(x-v)^2}{2}}$. Besides,

$$\begin{aligned}\int_0^{+\infty} e^{-u\frac{(x-v)^2}{2}}dx =& \sqrt{\frac{2}{u}}\int_0^{+\infty} e^{-\left(\sqrt{\frac{u}{2}}(x-v)\right)^2} d\Big(\sqrt{\frac{u}{2}}((x-v)\Big) \\ =& \sqrt{\frac{2}{u}}\int_{-v\sqrt{\frac{u}{2}}}^{+\infty} e^{-s^2}ds \leqslant \sqrt{\frac{2}{u}}\int_{-\infty}^{+\infty} e^{-s^2}ds = \sqrt{\frac{2\pi}{u}},\ \forall u,v>0.\end{aligned}$$

Hence, in view of the hypothesis $f(u) \in L_1\Big(R_+, \frac{1}{\sqrt{u^3}}\Big)$ and $g \in L_1\big(R_+\big)$, we have

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} e^{-ucosh(x-v)}\frac{|f(u)|}{u}\,|g(v)|dudvdx \\ \leqslant& \int_0^{+\infty}\int_0^{+\infty}\frac{\sqrt{2\pi}}{\sqrt{u}}\frac{|f(u)|}{u}|g(v)|dudv = \sqrt{2\pi}\int_0^{+\infty}\frac{|f(u)|}{\sqrt{u^3}}du\int_0^{+\infty}|g(v)|dv < +\infty. \qquad (3.4)\end{aligned}$$

Similarly,

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} e^{-ucosh(x+v)}\frac{|f(u)|}{u}\,|g(v)|dudvdx < +\infty. \qquad (3.5)$$

It follows from (3.1), (3.4) and (3.5) that

$$\int_0^{+\infty} \big|(f*g)(x)\big|dx \leqslant \frac{1}{\pi^2}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{1}{u}\big[e^{-ucosh(x-v)} + e^{-ucosh(x+v)}\big]|f(u)|\,|g(v)|dudvdx < +\infty. \tag{3.6}$$

Thus, $(f*g)(x)$ belongs to $L_1(R_+)$.

Now we prove the factorization property (3.2). We have

$$(Kf)(y)(F_sg)(y) = \sqrt{\frac{8}{\pi^5}}\int_0^{+\infty}\int_0^{+\infty} \frac{1}{u}\sin(yv)K_{iy}(u)f(u)g(v)dudv,\ y>0.$$

Using formula 1.98 in ([15], p. 14) we obtain

$$\begin{aligned}(Kf)(y)(F_sg)(y) &= \sqrt{\frac{8}{\pi^5}}\int_0^{+\infty}\int_0^{+\infty} \frac{1}{u}\sin(yv)f(u)g(v)\Big[\int_0^{+\infty}\cos(y\alpha)\ e^{-ucosh\alpha}d\alpha\Big]dudv \\ &= \sqrt{\frac{8}{\pi^5}}\int_0^{+\infty}\int_0^{+\infty} \frac{1}{u}f(u)g(v)\Big\{\int_0^{+\infty}\big[\sin y(\alpha+v) - \sin y(\alpha-v)\big]e^{-ucosh\alpha}d\alpha\Big\}dudv.\end{aligned} \tag{3.7}$$

Since,

$$\int_0^{+\infty} \sin y(\alpha+v)\ e^{-ucosh\alpha}d\alpha = \int_v^{+\infty}\sin(yt)\ e^{-ucosh(t-v)}dt, \tag{3.8}$$

$$\int_0^{+\infty} \sin y(\alpha-v)\ e^{-ucosh\alpha}d\alpha = \int_{-v}^{+\infty}\sin(yt)\ e^{-ucosh(t+v)}dt, \tag{3.9}$$

we have

$$\int_0^{+\infty}\big[\sin y(\alpha+v) - \sin y(\alpha-v)\big]e^{-ucosh\alpha}d\alpha \tag{3.10}$$

$$= \int_v^{+\infty}\sin(yt)\ e^{-ucosh(t-v)}dt - \int_{-v}^{0}\sin(yt)\ e^{-ucosh(t+v)}dt - \int_0^{+\infty}\sin(yt)\ e^{-ucosh(t+v)}dt \tag{3.11}$$

$$= \int_0^{+\infty}\sin(yt)\big[e^{-ucosh(t-v)} - e^{-ucosh(t+v)}\big]dt. \tag{3.12}$$

Using (3.7)-(3.12) we obtain

$$
\begin{aligned}
&(Kf)(y)(F_s g)(y)\\
&= \frac{1}{\pi^2}\sqrt{\frac{2}{\pi}}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{u}f(u)g(v)\Big\{\int_0^{+\infty}\sin(yt)\big[e^{-u\cosh(t-v)}-e^{-u\cosh(t+v)}\big]dt\Big\}dudv\\
&= \sqrt{\frac{2}{\pi}}\int_0^{+\infty}\sin(yt)\Big\{\frac{1}{\pi^2}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{u}f(u)g(v)\big[e^{-u\cosh(t-v)}-e^{-u\cosh(t+v)}\big]dudv\Big\}dt\\
&= F_s(f*g)(y).
\end{aligned}
$$

Thus, the factorization equality (3.2) is proved. ☐

Proposition 3.1. *For* $f(x) \in L_1\Big(R_+, \dfrac{1}{\sqrt{x^3}}\Big)$ *and* $g \in L_1\big(R_+\big)$*, the generalized convolution* (16) *can be represented in the form*

$$
(f*g)(x) = \frac{\sqrt{2}}{\pi\sqrt{\pi}}\int_0^{+\infty}\frac{1}{u}f(u)\big[(\operatorname{sign} v\, g(|v|) \underset{F}{*} e^{-u.\cosh v})(x)\big]du,\ x>0.
$$

Here the convolution $(\cdot \underset{F}{*} \cdot)$ *is defined by formula* (2.1).

Proof. Since $f(u) \in L_1\Big(R_+, \dfrac{1}{\sqrt{u^3}}\Big)$, from (3.1) we have

$$
\begin{aligned}
(f*g)(x) &= \frac{1}{\pi^2}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{u}\Big[e^{-u.\cosh(x-v)}-e^{-u.\cosh(x+v)}\Big]f(u)g(|v|)dudv\\
&= \frac{1}{\pi^2}\int_0^{+\infty}\frac{1}{u}f(u)\Big[\int_0^{+\infty}e^{-u.\cosh(x-v)}g(|v|)dv+\int_0^{-\infty}e^{-u\cosh(x-t)}g(|-t|)dt\Big]du\\
&= \frac{1}{\pi^2}\int_0^{+\infty}\frac{1}{u}f(u)\Big[\int_0^{+\infty}e^{-u.\cosh(x-v)}\operatorname{sign} v\, g(|v|)dv+\int_{-\infty}^{0}e^{-u.\cosh(x-v)}\operatorname{sign} v\, g(|v|)dv\Big]du\\
&= \frac{1}{\pi^2}\int_0^{+\infty}\frac{1}{u}f(u)\Big[\int_{-\infty}^{+\infty}e^{-u.\cosh(x-v)}\operatorname{sign} v\, g(|v|)dv\Big]du\\
&= \frac{\sqrt{2}}{\pi\sqrt{\pi}}\int_0^{+\infty}\frac{1}{u}f(u)\big[\operatorname{sign} v\, g(|v|) \underset{F}{*} e^{-u\cosh v})(x)\big]du.
\end{aligned}
$$

The proof is complete. ☐

Remark 3.2. *Since* $\operatorname{sign} v \cdot g(|v|)$ *is an odd function and* $e^{-ucoshv}$ *is an even function in* v, *using Proposition 1 we get the following indentity*

$$(f * g)(x) = \frac{\sqrt{2}}{\pi\sqrt{\pi}} \int_0^{+\infty} \frac{1}{u} f(u)\big[\big(g(v) \underset{1}{*} e^{-ucoshv}\big)(x)\big] du,\ x > 0,$$

where the convolution $(\cdot \underset{1}{*} \cdot)$ *is given by* (2.3).

Proposition 3.2. *For* $f, g \in L_1\Big(R_+, \dfrac{1+\sqrt{x^3}}{\sqrt{x^3}}\Big)$, *the generalized convolution* (3.1) *is not commutative. Moreover,*

$$(f * g)(x) = (g * f)(x) + \int_0^{+\infty} \frac{1}{u} \left(\begin{vmatrix} f(u) & f(v) \\ g(u) & g(v) \end{vmatrix} \underset{1}{*} e^{-ucoshv} \right)(x) du.$$

Proof. From the hypothesis, it is clear that $f, g \in L_1\Big(R_+, \dfrac{1}{\sqrt{x^3}}\Big)$ and $f, g \in L_1(R_+)$. Using the above remark we obtain

$$
\begin{aligned}
&(f{*}g)(x) - (g{*}f)(x) = \\
&= \frac{\sqrt{2}}{\pi\sqrt{\pi}} \Big[\int_0^{+\infty} \frac{1}{u} f(u)\big[\big(g(v) \underset{1}{*} e^{-ucoshv}\big)(x)\big] du - \int_0^{+\infty} \frac{1}{u} g(u)\big[\big(f(v) \underset{1}{*} e^{-ucoshv}\big)(x)\big] du \\
&= \frac{\sqrt{2}}{\pi\sqrt{\pi}} \int_0^{+\infty} \frac{1}{u} \Big[f(u)\big(g(v) \underset{1}{*} e^{-ucoshv}\big)(x) - g(u)\big(f(v) \underset{1}{*} e^{-ucoshv}\big)(x)\Big] du \\
&= \frac{\sqrt{2}}{\pi\sqrt{\pi}} \int_0^{+\infty} \frac{1}{u} \big[\big(f(u)g(v) \underset{1}{*} e^{-ucoshv}\big)(x) - \big(g(u)f(v) \underset{1}{*} e^{-ucoshv}\big)(x)\big] du \\
&= \frac{\sqrt{2}}{\pi\sqrt{\pi}} \int_0^{+\infty} \frac{1}{u} \big(\big(f(u)g(v) - g(u)f(v)\big) \underset{1}{*} e^{-ucoshv}\big)(x) dx \\
&= \frac{\sqrt{2}}{\pi\sqrt{\pi}} \int_0^{+\infty} \frac{1}{u} \left(\begin{vmatrix} f(u) & f(v) \\ g(u) & g(v) \end{vmatrix} \underset{1}{*} e^{-ucoshv} \right)(x) du.
\end{aligned}
$$

The proposition is proved. □

Proposition 3.3. *The generalized convolution* (3.1) *is not associative and satisfies the following equalities*

(a) $\quad f{*}(g \overset{\eta}{\underset{F_s}{*}} h) = (f{*}g) \overset{\eta}{\underset{F_s}{*}} h,$ *where* $f \in L_1\Big(R_+, \dfrac{1}{\sqrt{x^3}}\Big)$, $g, h \in L_1(R_+)$,

(b) $\quad g \overset{\eta}{\underset{F_s}{*}} (f{*}h) = (f{*}g) \overset{\eta}{\underset{F_s}{*}} h,$ *where* $f \in L_1\Big(R_+, \dfrac{1}{\sqrt{x^3}}\Big)$, $g, h \in L_1(R_+)$,

(c) $\quad (f*g) \underset{1}{*} h = f*(g \underset{1}{*} h),$ *where* $f \in L_1\Big(R_+, \dfrac{1}{\sqrt{x^3}}\Big)$, $g, h \in L_1(R_+)$,

(d) $\quad f*(g*h) = g*(f*h),$ *where* $f, g \in L_1\Big(R_+, \dfrac{1}{\sqrt{x^3}}\Big)$, $h \in L_1(R_+)$,

(e) $\quad f*\big(g \overset{\eta}{\underset{F_s}{*}} (h \overset{\gamma_1}{\underset{3}{*}} k)\big) = (f*g) \overset{\eta}{\underset{F_s}{*}} (h \overset{\gamma_1}{\underset{3}{*}} k),$ *where* $f \in L_1\Big(R_+, \dfrac{1}{\sqrt{x^3}}\Big)$, $h \in L_1\Big(R_+, \dfrac{1}{x}\Big)$, $g, k \in L_1(R_+)$,

(g) $\quad f*\big(g \underset{1}{*} (h \underset{2}{*} k)\big) = (f*g) \underset{1}{*} (h \underset{2}{*} k),$ *where* $f \in L_1\Big(R_+, \dfrac{1}{\sqrt{x^3}}\Big)$, $g, h, k \in L_1(R_+)$ *and the convolutions* $(\cdot \overset{\eta}{\underset{F_s}{*}} \cdot)$, $(\cdot \underset{1}{*} \cdot)$, $(\cdot \underset{2}{*} \cdot)$, $(\cdot \overset{\gamma_1}{\underset{3}{*}} \cdot)$ *are respectively defined by* (2.5), (2.3), (2.10) *and* (2.12).

Proof. a) From the factorization property

$$F_s(f*g)(y) = (Kf)(y)(F_s g)(y), \quad \forall y > 0,$$

we have

$$\begin{aligned}
&F_s\big(f*(g \overset{\eta}{\underset{F_s}{*}} h)\big)(y) = (Kf)(y)F_s(g \overset{\eta}{\underset{F_s}{*}} h)(y)\\
&= (Kf)(y)\eta(y)(F_s g)(y)(F_s h)(y)\\
&= \big((Kf)(y)(F_s g)(y)\big)\eta(y)(F_s h)(y)\\
&= F_s(f*g)(y)\eta(y)(F_s h)(y)\\
&= F_s\big((f*g)\big) \overset{\eta}{\underset{F_s}{*}} h\big)(y), \quad \forall y > 0.
\end{aligned}$$

This shows that

$$f * \big(g \overset{\eta}{\underset{F_s}{*}} h\big) = (f*g) \overset{\eta}{\underset{F_s}{*}} h.$$

The proofs of equalities (b), (c) and (d) are similar to that of (a).

e) From the factorization properties of the convolutions (3.2), (2.6) and (2.14) for $y > 0$ we have

$$\begin{aligned}
&F_s\big(f * \big(g \overset{\eta}{\underset{F_s}{*}} (h \overset{\gamma_1}{\underset{3}{*}} k)\big)\big)(y) = (Kf)(y)F_s\big(g \overset{\eta}{\underset{F_s}{*}} \big(h \overset{\gamma_1}{\underset{3}{*}} k\big)\big)(y)\\
&= \eta(y)(Kf)(y)(F_s g)(y)F_s(h \overset{\gamma_1}{\underset{3}{*}} k)(y)\\
&= \eta(y)F_s(f*g)(y)F_s(h \overset{\gamma_1}{\underset{3}{*}} k)(y)\\
&= F_s\big((f*g) \overset{\eta}{\underset{F_s}{*}} (h \overset{\gamma_1}{\underset{3}{*}})\big)(y).
\end{aligned}$$

Thus,

$$f * \big(g \overset{\eta}{\underset{F_s}{*}} (h \overset{\gamma_1}{\underset{3}{*}} k)\big) = (f*g) \overset{\eta}{\underset{F_s}{*}} (h \overset{\gamma_1}{\underset{3}{*}} k).$$

One can easily obtain part (e) in a similar way. ☐

Proposition 3.4. *There does not exist the unit element for the generalized convolution* (3.1).

Proof. We prove that there does not exist the left unit element for the generalized convolution (16). Suppose that there exists $e \in L_1\Big(R_+, \dfrac{1}{\sqrt{x^3}}\Big)$ such that

$$(e * f)(x) = f(x), \quad \forall f \in L_1(R_+).$$

From Theorem 3.1, we have

$$(F_s f)(y) = F_s(e * f)(y) = (Ke)(y)(F_s f)(y), \quad \forall y > 0.$$

Therefore

$$(F_s f)(y)\big[1 - (Ke)(y)\big] = 0.$$

This shows that

$$(Ke)(y) = 1.$$

Hence

$$e(y) = K^{-1}(1), \quad \forall y > 0. \tag{3.13}$$

Here K^{-1} denotes the inverse Kontorovich-Lebedev transform. From formula 9.7.4 in ([1], p. 199) the right hand side of (3.13) does not converge. Hence the equality (3.13) does not hold. This contradiction means the unit element of the generalized convolution (3.1) does not exist. The theorem is proved. □

4. Application to Solving Systems of Integral Equations

Not many systems of integral equations of the second kind can be solved in closed form. The generalized convolution (3.1) introduced in this paper allows us to get the solutions in closed form for two systems of integral equations.

a) Consider the system of integral equations

$$f(x) + \lambda_1 \int_0^{+\infty} \theta_1(x,u) g(u) du = h(x), \quad x > 0,$$

$$\lambda_2 \int_0^{+\infty} \theta_2(x,t) f(t) dt + g(x) = k(x), \quad x > 0. \tag{4.1}$$

Here

$$\theta_1(x,u) = \frac{1}{4\pi} \int_0^{+\infty} \frac{1}{u} \big[e^{-u\cosh(x-v)} - e^{-u\cosh(x+v)}\big] \varphi(v) dv;$$

and

$$\theta_2(x,t) = \frac{1}{2\pi} \int_0^{+\infty} \psi(u) \big[\operatorname{sign}(|x-t| - u)\, \xi\big(\big||x-t| - u\big|\big) + \xi(|x-t| + u)$$
$$- \operatorname{sign}(x+t-u)\, \xi\big(|x+t-u|\big) - \xi(x+t+u)\big]\, du;$$

$\varphi \in L_1\big(R_+, \frac{1}{\sqrt{x^3}}\big)$, $k, \psi, h, \xi \in L_1(R_+)$ are given; λ_1, λ_2 are complex constants; and f, g are unknown functions.

Theorem 4.1. *With the condition*

$$1-\lambda_1\lambda_2 F_c\big((\varphi*\psi)\underset{2}{*}\xi\big)(y)\neq 0,\quad \forall y>0, \tag{4.2}$$

the system (3.13) *has a unique solution in* $L_1(R_+)$ *which is represented by*

$$f(x)=h(x)-\lambda_1(\varphi*k)(x)+(h\underset{1}{*}q)(x)-\lambda_1\big((\varphi*k)\underset{1}{*}q\big)(x), \tag{4.3}$$

$$g(x)=k(x)-\lambda_2\big(\psi\underset{1}{*}(\xi\underset{2}{*}h)\big)(x)+(k\underset{1}{*}q)(x)-\lambda_2\big((\psi\overset{1}{*}(\xi\underset{2}{*}h))\underset{1}{*}q\big)(x). \tag{4.4}$$

Here, $q\in L_1(R_+)$ *is defined by*

$$(F_cq)(y)=\frac{\lambda_1\lambda_2F_c\big((\varphi*\psi)\underset{2}{*}\xi\big)(y)}{1-\lambda_1\lambda_2F_c\big((\varphi*\psi)\underset{2}{*}\xi\big)(y)},\quad \forall y>0,$$

and the convolutions $(\cdot\underset{1}{*}\cdot)$, $(\cdot\underset{2}{*}\cdot)$, $(\cdot*\cdot)$ *are defined respectively by* (2.3), (2.10), (3.1).

Proof. System (4.1) can be rewritten in the form

$$f(x)+\lambda_1(\varphi*g)(x)=h(x),$$
$$\lambda_2\big(f\underset{1}{*}(\psi\underset{2}{*}\xi)\big)(x)+g(x)=k(x),\quad x>0.$$

Using the factorization properties (2.4), (2.11), (3.2) we have

$$(F_sf)(y)+\lambda_1(K\varphi)(y)(F_sg)(y)=(F_sh)(y),$$
$$\lambda_2(F_sf)(y)(F_s\psi)(y)(F_s\xi)(y)+(F_sg)(y)=(F_sk)(y).$$

To solve for $(F_sf)(y)$ and $(F_s(g)(y)$ from this system we note that

$$\Delta=\begin{vmatrix}1 & \lambda_1(K\varphi)(y)\\ \lambda_2(F_s\psi)(y)(F_s\xi)(y) & 1\end{vmatrix}$$
$$=1-\lambda_1\lambda_2F_s(\varphi*\psi)(y)(F_s\xi)(y)=1-\lambda_1\lambda_2F_c\big((\varphi*\psi)\underset{2}{*}\xi\big)(y)\neq 0,\quad \forall y>0,$$

then

$$\frac{1}{\Delta}=1+\frac{\lambda_1\lambda_2F_c\big((\varphi*\psi)\underset{2}{*}\xi\big)(y)}{1-\lambda_1\lambda_2F_c\big((\varphi*\psi)\underset{2}{*}\xi\big)(y)}.$$

In virtue of Wiener-Lévy's theorem [2], from condition (28) there is a function $q(x)\in L_1(R_+)$ such that

$$\frac{\lambda_1\lambda_2F_c\big((\varphi*\psi)\underset{2}{*}\xi\big)(y)}{1-\lambda_1\lambda_2F_c\big((\varphi*\psi)\underset{2}{*}\xi\big)(y)}=(F_cq)(y).$$

Thus,

$$\frac{1}{\Delta}=1+(F_cq)(y),$$

and

$$\Delta_1=\begin{vmatrix}(F_sh)(y) & \lambda_1(K\varphi)(y)\\ (F_sk)(y) & 1\end{vmatrix}=(F_sh)(y)-\lambda_1F_s(\varphi*k)(y).$$

Therefore

$$\begin{aligned}(F_sf)(y) &= \frac{1}{\Delta}\Delta_1 = \big[1+(F_cq)(y)\big]\big[(F_sh)(y)-\lambda_1F_s(\varphi*k)(y)\big]\\ &= (F_sh)(y)-\lambda_1F_s(\varphi*k)(y)+(F_cq)(y)(F_sh)(y)\\ &\quad -\lambda_1(F_cq)(y)F_s(\varphi*k)(y),\ \forall y>0.\end{aligned}$$

It follows that

$$(F_sf)(y) = (F_sh)(y)-\lambda_1F_s(\varphi*k)(y)+F_s(h\underset{1}{*}q)(y)-\lambda_1F_s\big((\varphi*k)\underset{1}{*}q\big)(y),\ \forall y>0.$$

Hence

$$f(x) = h(x)-\lambda_1(\varphi*k)(x)+(h\underset{1}{*}q)(x)-\lambda_1\big((\varphi*k)\underset{1}{*}q\big)(x)\in L_1(R_+).$$

Similarly,

$$\begin{aligned}\Delta_2 = \begin{vmatrix}1 & (F_sh)(y)\\ \lambda_2(F_s\psi)(y)(F_s\xi)(y) & (F_sk)(y)\end{vmatrix} &= (F_sk)(y)-\lambda_2(F_s\psi)(y)F_c(\xi\underset{2}{*}h)(y)\\ &= (F_sk)(y)-\lambda_2F_s\big(\psi\underset{1}{*}(\xi\underset{2}{*}h)\big)(y).\end{aligned}$$

Therefore

$$\begin{aligned}(F_sg)(y) &= \frac{1}{\Delta}\Delta_2 = \big[1+(F_cq)(y)\big]\big[(F_sk)(y)-\lambda_2F_s\big(\psi\underset{1}{*}(\xi\underset{2}{*}h)\big)(y)\big]\\ &= (F_sk)(y)-\lambda_2F_s\big(\psi\underset{1}{*}(\xi\underset{2}{*}h)\big)(y)+(F_cq)(y)(F_sk)(y)-\lambda_2(F_cq)(y)F_s(\psi\underset{1}{*}(\xi\underset{2}{*}h)\big)(y)\\ &= (F_sk)(y)-\lambda_2F_s\big(\psi\underset{1}{*}(\xi\underset{2}{*}h)\big)(y)+F_s(k\underset{1}{*}q)(y)-\lambda_2F_s\big(((\psi\underset{1}{*}(\xi\underset{2}{*}h))\underset{1}{*}q\big)(y),\ \forall y>0.\end{aligned}$$

Hence

$$g(x) = k(x)-\lambda_2\big(\psi\underset{1}{*}(\xi\underset{2}{*}h)\big)(x)+(k\underset{1}{*}q)(x)-\lambda_2\big((\psi\underset{1}{*}(\xi\underset{2}{*}h)\big)\underset{1}{*}q\big)(x)\in L_1(R_+).$$

One can easily verify that f and g given by (4.3), (4.4) satisfy the system (4.1). The proof is complete. □

b) Consider the system of integral equations

$$\begin{aligned} f(x)+\lambda_1\int_0^{+\infty}\theta_1(x,u)g(u)du &= k(x),\quad x>0,\\ \lambda_3\int_0^{+\infty}\theta_3(x,v)f(v)dv+g(x) &= k(x),\quad x>0. \end{aligned}\tag{4.5}$$

Here

$$\theta_1(x,u) = \frac{1}{4\pi}\int_0^{+\infty}\frac{1}{u}\big[e^{-ucosh(x-v)}-e^{-ucosh(x+v)}\big]\varphi(v)dv;$$

and

$$\theta_3(x,v) = \frac{1}{\pi^2\sqrt{2\pi}} \int_0^{+\infty}\int_0^{+\infty} \psi(u)\big\{\big[sinh(x+t)e^{-ucosh(x+t)} + sinh(x-t)e^{-ucosh(x-t)}\big] \times \big[\mathrm{sign}(v-t)\xi(|v-t|) + \xi(v+t)\big]\big\}dudt, \quad x>0;$$

$\varphi \in L_1\Big(R_+, \dfrac{1}{\sqrt{x^3}}\Big)$, $\psi \in L_1\Big(R_+, \dfrac{1}{x}\Big)$, $h,k,\xi \in L_1(R_+)$ are given; λ_1, λ_2 are complex constants; and f, g are unknown functions.

Theorem 4.2. *With the condition*

$$1 - \lambda_1\lambda_2 F_c\big(\psi \overset{\gamma_1}{\underset{4}{*}} (\varphi * \xi)\big)(y) \neq 0,\ \forall y>0, \tag{4.6}$$

the system (4.5) *has the unique solution in* $L_1(R_+)$*:*

$$f(x) = h(x) - \lambda_1(\varphi * k)(x) + (h \underset{1}{*} l)(x) - \lambda_1\big((\varphi * k) \underset{1}{*} l\big)(x), \tag{4.7}$$

$$g(x) = k(x) - \lambda_2\big(\psi \overset{\gamma_1}{\underset{3}{*}} (h \underset{2}{*} \xi)\big)(x) + (k \underset{1}{*} l)(x) - \lambda_2\big((\psi \overset{\gamma_1}{\underset{3}{*}} (h \underset{2}{*} \xi) \underset{1}{*} l\big)(x). \tag{4.8}$$

Here, $l \in L_1(R_+)$ *is defined by*

$$\frac{\lambda_1\lambda_2 F_c\big(\psi \overset{\gamma_1}{\underset{4}{*}} (\varphi * \xi)\big)(y)}{1 - \lambda_1\lambda_2 F_c\big(\psi \overset{\gamma_1}{\underset{4}{*}} (\varphi * \xi)\big)(y)} = (F_c l)(y),$$

where the convolutions $(\cdot * \cdot)$, $(\cdot \underset{1}{*} \cdot)$, $(\cdot \overset{\gamma_1}{\underset{3}{*}} \cdot)$, $(\cdot \overset{\gamma_1}{\underset{4}{*}} \cdot)$ *and* $(\cdot \underset{2}{*} \cdot)$ *are defined respectively by* (3.1), (2.3), (2.13) *and* (2.11).

Proof. System (4.5) can be rewritten in the following form

$$f(x) + \lambda_1(\varphi * g)(x) = h(x),$$
$$\lambda_2\big(\psi \overset{\gamma_1}{\underset{3}{*}} (f \underset{2}{*} \xi)\big)(x) + g(x) = k(x), \quad x>0.$$

Using the factorization properties (3.2), (2.14), (2.11) and (2.15) we get

$$(F_s f)(y) + \lambda_1(K\varphi)(y)(F_s g)(y) = (F_s h)(y),$$
$$\lambda_2\gamma_1(y)(K\psi)(y)F_c(f \underset{2}{*} \xi)(y) + (F_s g)(y) = (F_s k)(y), \quad \forall y>0. \tag{4.9}$$

To solve for $(F_s f)(y)$ and $(F_s g)(y)$ from this system we note that

$$\begin{aligned}\Delta =& \begin{vmatrix} 1 & \lambda_1(K\varphi)(y) \\ \lambda_2\gamma_1(y)(K^{-1}\psi)(y)(F_s\xi)(y) & 1 \end{vmatrix} \\ =& \lambda_1\lambda_2\gamma_1(y)(K\psi)(y)F_s(\varphi * \xi)(y) \\ =& 1 - \lambda_1\lambda_2 F_c\big(\psi \overset{\gamma_1}{\underset{4}{*}} (\varphi * \xi)\big)(y) \neq 0, \quad \forall y>0,\end{aligned}$$

$$\frac{1}{\Delta} = 1 + \frac{\lambda_1\lambda_2 F_c\big(\psi \overset{\gamma_1}{\underset{4}{*}} (\varphi * \xi)\big)(y)}{1 - \lambda_1\lambda_2 F_c\big(\psi \overset{\gamma_1}{\underset{4}{*}} (\varphi * \xi)\big)(y)}.$$

In virtue of Wiener-Lévy's theorem [2], from condition (4.6) there is a unique function $l(x) \in L_1(R_+)$ such that

$$\frac{\lambda_1\lambda_2 F_c\big(\psi \overset{\gamma_1}{\underset{4}{*}} (\varphi * \xi)\big)(y)}{1-\lambda_1\lambda_2 F_c\big(\psi \overset{\gamma_1}{\underset{4}{*}} (\varphi * \xi)\big)(y)} = (F_c l)(y).$$

Thus,

$$\frac{1}{\Delta} = 1 + (F_c l)(y).$$

Solving the linear system (4.9) we obtain

$$\begin{aligned}(F_s f)(y) &= \big[1+(F_c l)(y)\big]\big[(F_s h)(y) - \lambda_1 F_s(\varphi * k)(y)\big]\\ &= (F_s h)(y) - \lambda_1 F_s(\varphi * k)(y) + (F_c l)(F_s h)(y) - \lambda_1 (F_c l)(y) F_s(\varphi * k)(y)\\ &= (F_s h)(y) - \lambda_1 F_s(\varphi * k)(y) + F_s(h \underset{1}{*} l)(y) - \lambda_1 F_s\big((\varphi * k) \underset{1}{*} l\big)(y),\ \forall y>0,\end{aligned}$$

and

$$\begin{aligned}(F_s g)(y) &= \big[1+(F_c l)(y)\big]\big[(F_s k)(y) - \lambda_2 F_s\big(\psi \overset{\gamma_1}{\underset{3}{*}} (h \underset{2}{*} \xi)\big)(y)\big]\\ &= (F_s k)(y) - \lambda_2 F_s\big(\psi \overset{\gamma_1}{\underset{3}{*}} (h \underset{2}{*} \xi)\big)(y) + (F_c l)(F_s k)(y)\\ &\quad - \lambda_2 (F_c l)(y) F_s\big(\psi \overset{\gamma_1}{\underset{3}{*}} (h \underset{2}{*} \xi)\big)(y)\\ &= (F_s k)(y) - \lambda_2 F_s\big(\psi \overset{\gamma_1}{\underset{3}{*}} (h \underset{2}{*} \xi)\big)(y) + F_s(h \underset{1}{*} l)(y)\\ &\quad - \lambda_2 F_s\big(\big(\psi \overset{\gamma_1}{\underset{3}{*}} (h \underset{2}{*} \xi)\big) \underset{1}{*} l\big)(y), \quad \forall y>0.\end{aligned}$$

Hence

$$\begin{aligned}f(x) =& h(x) - \lambda(\varphi * k)(x) + (h \underset{1}{*} l)(x) - \lambda_1\big((\varphi * k) \underset{1}{*} l\big)(x) \in L_1(R_+),\\ g(x) =& k(x) - \lambda_2\big(\psi \overset{\gamma_1}{\underset{3}{*}} (h \underset{2}{*} \xi)\big)(x) + (k \underset{1}{*} l)(x) - \lambda_2\big(\big(\psi \overset{\gamma_1}{\underset{3}{*}} (h \underset{2}{*} \xi)\big) \underset{1}{*} l\big)(x) \in L(R_+).\end{aligned}$$

One can easily verify that f and g given by (4.7), (4.8) satisfy the system (4.5). The theorem is proved. □

Acknowledgments

The authors would like to express their deep thank to Prof. Dr. Vu Kim Tuan for his valuable suggestions.

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