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MAPPINGS IN σ-PONOMAREV-SYSTEMS

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Abstract. We use the σ-Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$ to give a consistent method to construct an s-mapping ($msss$-mapping, $mssc$-mapping) f with covering-properties onto a space X from a metric space M. As applications, we systematically get characterizations of s-images ($msss$-images, $mssc$-images) of metric spaces.

1. Introduction

In [22], S. Lin and P. Yan introduced Ponomarev-systems $(f, M, X, \mathcal{P})$ and $(f, M, X, \{\mathcal{P}_n\})$ to establish the general condition to construct a compact-covering mapping f onto a space X from some metric space M. After that, these notions were investigated in [9], [10], [11], [12], [28], and necessary and sufficient conditions such that f is an s-mapping with covering-properties have been stated. Among mappings with metric domains, $msss$-mappings and $mssc$-mappings play important roles, and these mappings cause attentions in [4], [8], [17], [19]. By definitions of mappings, we have that

$$mssc\text{-}mapping \Rightarrow msss\text{-}mapping \Rightarrow s\text{-}mapping.$$

However, for Ponomarev-systems $(f, M, X, \mathcal{P})$ and $(f, M, X, \{\mathcal{P}_n\})$, we do not know what necessary and sufficient conditions such that the mapping f is an $msss$-mapping ($mssc$-mapping) with covering-properties onto X from a metric space M are. So, we are interested in finding a consistent method to construct an s-mapping ($msss$-mapping, $mssc$-mapping) with covering-properties from some metric space M onto a space X.

In this paper, we use the σ-Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$ to give a consistent method to construct an s-mapping ($msss$-mapping, $mssc$-mapping) f with covering-properties onto X from a metric space M. As applications, we systematically get characterizations of s-images ($msss$-images, $mssc$-images) of metric spaces. These results make the study of images of metric spaces more completely.

The paper is organized as follows. Beside the introduction, the paper contains two sections. In Section 2 we introduce definitions and lemmas which will be used throughout the paper. The main results are presented in Section 3.

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2. Preliminaries

Throughout this paper, all spaces are T_1 and regular, all mappings are continuous and onto, a convergent sequence includes its limit point, $\mathbb{N}$ denotes the set of all natural numbers, $\omega = \mathbb{N} \cup \{0\}$, and p_k denotes the projection of $\prod_{n\in\mathbb{N}} X_n$ onto X_k. Let $f : X \longrightarrow Y$ be a mapping, $x \in X$, and $\mathcal{P}$ be a family of subsets of X, we denote $st(x, \mathcal{P}) = \bigcup\{P \in \mathcal{P} : x \in P\}$, $\bigcup \mathcal{P} = \bigcup\{P : P \in \mathcal{P}\}$, $\bigcap \mathcal{P} = \bigcap\{P : P \in \mathcal{P}\}$, and $f(\mathcal{P}) = \{f(P) : P \in \mathcal{P}\}$. We say that a convergent sequence $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ converging to x is *eventually* (resp., *frequently*) in A if $\{x_n : n \geq n_0\} \cup \{x\} \subset A$ for some $n_0 \in \mathbb{N}$ (resp., $\{x_{n_k} : k \in \mathbb{N}\} \cup \{x\} \subset A$ for some subsequence $\{x_{n_k} : k \in \mathbb{N}\}$ of $\{x_n : n \in \mathbb{N}\}$).

Definition 2.1. Let $\mathcal{P}$ be a cover for a space X and K be a subset of X.

(1) $\mathcal{P}$ is a *network for K in X*, if $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in K\}$, where $x \in \bigcap \mathcal{P}_x$, and whenever $x \in U$ with U open in X, there exists $P \in \mathcal{P}_x$ such that $x \in P \subset U$ for every $x \in K$. Here, $\mathcal{P}_x$ is a *network at x in K*. If $K = X$, then a network for K in X is a *network for X* [23], and a network at x in K is a *network at x in X*.

(2) $\mathcal{P}$ is a *cfp-network for K in X* [1], if for each compact subset H of K and $H \subset U$ with U open in X, there exists a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ such that $H \subset \bigcup\{C_F : F \in \mathcal{F}\} \subset \bigcup \mathcal{F} \subset U$, where C_F is closed and $C_F \subset F$ for every $F \in \mathcal{F}$. If $K = X$, then a cfp-network for K in X is a *cfp-network for X* [30].

(3) $\mathcal{P}$ is a *cs-network for K in X* (resp., *cs^*-network for K in X*) [1], if for each convergent sequence S in K converging to $x \in U$ with U open in X, S is eventually (resp., frequently) in $P \subset U$ with some $P \in \mathcal{P}$. If $K = X$, then a cs-network for K in X (resp., cs^*-network for K in X) is a *cs-network for X* [14] (resp., *cs^*-network for X* [6]).

(4) $\mathcal{P}$ is a *wcs-network for K in X*, if for each convergent sequence S in K converging to $x \in U$ with U open in X, S is eventually in $\bigcup \mathcal{F} \subset U$ with some finite subfamily $\mathcal{F}$ of $\{P \in \mathcal{P} : x \in P\}$. If $K = X$, then a wcs-network for K in X is a *wcs-network for X*.

(5) $\mathcal{P}$ is *point-countable* [13], if for each $x \in X$, $\{P \in \mathcal{P} : x \in P\}$ is countable.

Remark 2.2. (1) A network (resp., cfp-network, cs-network, cs^*-network, wcs-network) for X is abbreviated to a network (resp., cfp-network, cs-network, cs^*-network, wcs-network).

(2) A countable cfp-network, a countable cs^*-network, and a countable wcs-network for a convergent sequence are equivalent.

Definition 2.3. Let X be a space.

(1) X is a *cosmic* space [24] (resp., *$\aleph_0$-space* [24], *$\aleph$-space* [25]), if X has a countable network (resp., countable cs-network, σ-locally finite cs-network).

(2) A subset P of X is *relatively compact* in X, if $\overline{P}$ is a compact subset of X.

Definition 2.4. Let $f : X \longrightarrow Y$ be a mapping.

(1) f is a *metrizable stratified strong separable* mapping or an *$msss$-mapping* [19], if X is a subspace of the product space $\prod_{n\in\mathbb{N}} X_n$ of a family $\{X_n : n \in \mathbb{N}\}$

of metric spaces, and for each $y \in Y$, there exists a sequence $\{V_{y,n} : n \in \mathbb{N}\}$ of open neighborhoods of y in Y such that each $p_n(f^{-1}(V_{y,n}))$ is a separable subset of X_n.

(2) f is a *metrizable stratified strong compact* mapping or an *mssc-mapping* [19], if X is a subspace of the product space $\prod_{n\in\mathbb{N}} X_n$ of a family $\{X_n : n \in \mathbb{N}\}$ of metric spaces, and for each $y \in Y$, there exists a sequence $\{V_{y,n} : n \in \mathbb{N}\}$ of open neighborhoods of y in Y such that each $\overline{p_n(f^{-1}(V_{y,n}))}$ is a compact subset of X_n. Moreover, if X is a relatively compact subset of $\prod_{n\in\mathbb{N}} X_n$, then f is a *relatively compact-metrizable stratified strong compact* mapping or a *rc-mssc-mapping*.

(3) f is a *sequence-covering* mapping [26] if, for each convergent sequence S of Y, there exists a convergent sequence L of X such that $f(L) = S$. Note that a sequence-covering mapping is a *strong sequence-covering* mapping in the sense of [17].

(4) f is a *compact-covering* mapping [24] if, for each compact subset K of Y, there exists a compact subset L of X such that $f(L) = K$.

(5) f is a *pseudo-sequence-covering* mapping [15] if, for each convergent sequence S of Y, there exists a compact subset K of X such that $f(K) = S$. Note that a pseudo-sequence-covering mapping is a *sequence-covering* mapping in the sense of [13].

(6) f is a *subsequence-covering* mapping [21] if, for each convergent sequence S of Y, there exists a compact subset K of X such that $f(K)$ is a subsequence of S.

(7) f is a *sequentially-quotient* mapping or a *sequentially quotient, sequentially continuous* mapping in the sense of [3] if, for each convergent sequence S of Y, there exists a convergent sequence L of X such that $f(L)$ is a subsequence of S.

(8) f is an *s-mapping* [2], if for each $y \in Y$, $f^{-1}(y)$ is a separable subset of X.

Definition 2.5. Let $\mathcal{P}$ be a cover for a space X.

(1) $\mathcal{P}$ is a *strong network* for X if, for each $x \in X$, there exists a countable $\mathcal{P}_x \subset \mathcal{P}$ such that $\mathcal{P}_x$ is a network at x in X.

(2) $\mathcal{P}$ is a *strong cs-network* for X if, for each convergent sequence S of X, there exists a countable $\mathcal{P}_S \subset \mathcal{P}$ such that $\mathcal{P}_S$ is a *cs*-network for S in X.

(3) $\mathcal{P}$ is a *strong cs*-network* for X if, for each convergent sequence S of X, there exists a countable $\mathcal{P}_S \subset \mathcal{P}$ such that $\mathcal{P}_S$ is a *cs**-network for some subsequence of S in X.

(4) $\mathcal{P}$ is a *strong cfp-network* for X if, for each compact subset K of X, there exists a countable $\mathcal{P}_K \subset \mathcal{P}$ such that $\mathcal{P}_K$ is a *cfp*-network for K in X.

(5) $\mathcal{P}$ is a *strong wcs-network* for X if, for each convergent sequence S of X, there exists a countable $\mathcal{P}_S \subset \mathcal{P}$ such that $\mathcal{P}_S$ is a *wcs*-network for S in X.

(6) $\mathcal{P}$ is a *σ-strong network for X* [15], if $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ satisfying that each $\mathcal{P}_n$ is a cover for X, $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$, and $\{st(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ is a network at x in X for every $x \in X$.

For terms which are not defined here, please refer to [5] and [27].

Lemma 2.6. *If $\mathcal{P}$ is a cs-network for a convergent sequence $S \subset U$ with U open in a space X, then there exists $\mathcal{F} \subset \mathcal{P}$ satisfying the following:*

(1) *$\mathcal{F}$ is finite;*
(2) *For each $F \in \mathcal{F}$, $\emptyset \neq F \cap S \subset F \subset U$;*
(3) *For each $x \in S$, there exists a unique $F \in \mathcal{F}$ such that $x \in F$;*
(4) *If $F \in \mathcal{F}$ contains the limit point of S, then $S - F$ is finite.*

Such an $\mathcal{F}$ is called to have property $cs(S, U)$.

Proof. Let $S = \{x_n : n \in \omega\}$ with the limit point x_0. Since $\mathcal{P}$ is a cs-network for S in X, there exists some $P_0 \in \mathcal{P}$ such that S is eventually in $P_0 \subset U$. Then $S - P_0$ is finite. For each $x \in S - P_0$, there exists some $P_x \in \mathcal{P}$ such that $x \in P_x \subset U \cap (X - (S - \{x\}))$. Put $\mathcal{F} = \{P_0\} \cup \{P_x : x \in S - P_0\}$. It is easy to see that $\mathcal{F}$ has property $cs(S, U)$. □

Lemma 2.7. *If $\mathcal{P}$ is a cfp-network for a compact subset $K \subset U$ with U open in a space X, then there exists $\mathcal{F} \subset \mathcal{P}$ satisfying the following:*

(1) *$\mathcal{F}$ is finite;*
(2) *For each $F \in \mathcal{F}$, $\emptyset \neq F \cap K \subset F \subset U$;*
(3) *For each $F \in \mathcal{F}$, $\mathcal{F} - \{F\}$ is not a cover for K;*
(4) *For each $F \in \mathcal{F}$, $F \cap K$ is compact.*

Such an $\mathcal{F}$ is called to have property $cfp(K, U)$.

Proof. Since $\mathcal{P}$ is a cfp-network for K in X, then there exists a finite $\mathcal{Q} \subset \mathcal{P}$ such that $K \subset \bigcup\{C_Q : Q \in \mathcal{Q}\} \subset \bigcup \mathcal{Q} \subset U$, where $C_Q \subset Q$ is closed for every $Q \in \mathcal{Q}$. It is easy to pick $\mathcal{F} \subset \mathcal{Q}$ such that $\mathcal{F}$ has property $cfp(K, U)$. □

Definition 2.8. Let $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a network for a space X such that for each $x \in X$, there exists a network $\{P_{\alpha_n} : n \in \mathbb{N}\}$ at x in X with each $P_{\alpha_n} \in \mathcal{P}_n$. We may assume that $X \in \mathcal{P}_n \subset \mathcal{P}_{n+1}$ for each $n \in \mathbb{N}$ and $\mathcal{P}$ is closed under finite intersections, if necessary. Let $\mathcal{P}_n = \{P_\alpha : \alpha \in A_n\}$, where each A_n endowed with the discrete topology, then A_n is a metric space. Put

$$M = \big\{a = (\alpha_n) \in \prod_{n \in \mathbb{N}} A_n : \{P_{\alpha_n} : n \in \mathbb{N}\} \text{ forms a network at some point } x_a \text{ in } X\big\}.$$

Then, M is a metric space, and x_a is unique for each $a \in M$. Define $f : M \longrightarrow X$ by $f(a) = x_a$ for every $a \in M$. Then f is a mapping by the following Lemma 2.9. The system $(f, M, X, \{\mathcal{P}_n\})$ is a *σ-Ponomarev-system.*

Lemma 2.9. *Let $(f, M, X, \{\mathcal{P}_n\})$ be the system in Definition 2.8. Then the following hold.*

(1) *f is onto.*
(2) *f is continuous.*

Proof. (1). It is obvious.

(2). For each $a = (\alpha_n) \in M$ and $f(a) = x_a$, let V be an open neighborhood of x_a in X. Then there exists $k \in \mathbb{N}$ such that $x_a \in P_{\alpha_k} \subset V$. Put $U = \{b = (\beta_n) \in M : \beta_k = \alpha_k\}$. Then U is an open neighborhood of a in M and $f(U) \subset V$. This implies that f is continuous. □

Remark 2.10. (1) In [22], the Ponomarev-system $(f, M, X, \mathcal{P})$ requires $\mathcal{P}$ being a strong network for X, and the Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$ requires $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ being a σ-strong network for X.

(2) For a σ-Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$, we have that $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ can not be a σ-strong network for X whenever the topology on X is not trivial by the assumption $X \in \mathcal{P}_n \subset \mathcal{P}_{n+1}$, and if $\mathcal{P}_n = \mathcal{P}$ for every $n \in \mathbb{N}$, then $(f, M, X, \{\mathcal{P}_n\})$ is a Ponomarev-system $(f, M, X, \mathcal{P})$ in the sense of [22].

Lemma 2.11. *Let $(f, M, X, \{\mathcal{P}_n\})$ be a σ-Ponomarev-system, $a = (\alpha_n) \in M$ where $\{P_{\alpha_n} : n \in \mathbb{N}\}$ is a network at some point x_a in X, and*

$$U_n = \{b = (\beta_i) \in M : \beta_i = \alpha_i \text{ if } i \leq n\},$$

for every $n \in \mathbb{N}$. Then the following hold.

(1) *$\{U_n : n \in \mathbb{N}\}$ is a base at a in M.*
(2) *$f(U_n) = \bigcap_{i=1}^{n} P_{\alpha_i}$ for every $n \in \mathbb{N}$.*

Proof. (1). It is obvious.

(2). For each $n \in \mathbb{N}$, let $x \in f(U_n)$. Then $x = f(b)$ for some $b = (\beta_i) \in U_n$. This implies that $x = \bigcap_{i \in \mathbb{N}} P_{\beta_i} \subset \bigcap_{i=1}^{n} P_{\beta_i} = \bigcap_{i=1}^{n} P_{\alpha_i}$. Then $f(U_n) \subset \bigcap_{i=1}^{n} P_{\alpha_i}$.

Conversely, let $x \in \bigcap_{i=1}^{n} P_{\alpha_i}$, where $x = f(b)$ for some $b = (\beta_i) \in M$. For each $i \in \mathbb{N}$, since $\mathcal{P}_i \subset \mathcal{P}_{n+i}$, there exists $\gamma_{n+i} \in A_{n+i}$ such that $\gamma_{n+i} = \beta_i$. Put $c = (\gamma_i)$, where $\gamma_i = \alpha_i$ for all $i \leq n$. Then we get $c \in U_n$ and $f(c) = x$. This implies that $\bigcap_{i=1}^{n} P_{\alpha_i} \subset f(U_n)$.

By the above inclusion, we get $f(U_n) = \bigcap_{i=1}^{n} P_{\alpha_i}$. □

3. Main results

In [9] and [11], Y. Ge and S. Lin stated necessary and sufficient conditions such that f is an s-mapping for Ponomarev-systems $(f, M, X, \mathcal{P})$ and $(f, M, X, \{\mathcal{P}_n\})$. But we do not know whether conditions for *msss*-mappings and *mssc*-mappings can be obtained in these systems. In the following, we state necessary and sufficient conditions such that f is an s-mapping (*mssc*-mapping, *msss*-mapping) for a σ-Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$.

Theorem 3.1. *Let $(f, M, X, \{\mathcal{P}_n\})$ be a σ-Ponomarev-system. Then the following hold.*

(1) *f is an s-mapping if and only if each $\mathcal{P}_n$ is point-countable.*
(2) *f is an msss-mapping if and only if each $\mathcal{P}_n$ is locally countable.*
(3) *f is an mssc-mapping if and only if each $\mathcal{P}_n$ is locally finite.*

Proof. (1). *Necessity.* Let f be an s-mapping. If there exists $k \in \mathbb{N}$ such that $\mathcal{P}_k$ is not point-countable, then, for some $x \in X$, we have that $A_{x,k} = \{\alpha \in A_k : x \in P_\alpha\}$ is uncountable. For each $\alpha \in A_{x,k}$, put $U_\alpha = \{c = (\gamma_n) \in M : \gamma_k = \alpha\}$, then U_α is open. If $c = (\gamma_n) \in f^{-1}(x)$, then $x = f(c) \in P_{\gamma_k}$. This implies that $\gamma_k = \alpha$ for some $\alpha \in A_{x,k}$, hence $c \in U_\alpha$. Therefore, $\{U_\alpha : \alpha \in A_{x,k}\}$ is an uncountable open cover for $f^{-1}(x)$, but it has not any proper subcover. So $f^{-1}(x)$ is not separable, hence f is not an s-mapping. This is a contradiction.

Sufficiency. Let each $\mathcal{P}_n$ be point-countable. For each $x \in X$, we have that $A_{x,n} = \{\alpha \in A_n : x \in P_\alpha\}$ is countable for every $n \in \mathbb{N}$. Then $\prod_{n\in\mathbb{N}} A_{x,n}$ is hereditarily separable. It follows from $f^{-1}(x) \subset \prod_{n\in\mathbb{N}} A_{x,n}$ that $f^{-1}(x)$ is separable. Then f is an s-mapping.

(2). *Necessity.* Let f be an $msss$-mapping. If there exists $k \in \mathbb{N}$ such that $\mathcal{P}_k$ is not locally countable, then, for some $x \in X$, we have that $A_{x,k} = \{\alpha \in A_k : P_\alpha \cap U_x \neq \emptyset\}$ is uncountable for every open neighborhood U_x of x in X. For each $\alpha \in A_{x,k}$, pick some $y \in P_\alpha \cap U_x$, and put $y = f(a)$ with $a = (\alpha_n) \in M$. Put $b_\alpha = (\beta_n)$, where $\beta_n = \alpha_n$ if $n < k$, $\beta_k = \alpha$, and $\beta_n = \alpha_{n-1}$ if $n > k$. Then $\beta_n \in A_n$ for every $n \in \mathbb{N}$ by $\mathcal{P}_n \subset \mathcal{P}_{n+1}$, and $\{P_{\beta_n} : n \in \mathbb{N}\}$ forms a network at y in X. So $b_\alpha \in f^{-1}(y) \subset f^{-1}(U_x)$. This implies that $\alpha = p_k(b_\alpha) \in p_k(f^{-1}(U_x))$. Then $A_{x,k} \subset p_k(f^{-1}(U_x)) \subset A_k$. Since $A_{x,k}$ is uncountable and A_k is discrete, $p_k(f^{-1}(U_x))$ is not separable. This is a contradiction to the fact that f is an $msss$-mapping.

Sufficiency. Let each $\mathcal{P}_n$ be locally countable. For each $x \in X$, there exists an open neighborhood $U_{x,n}$ of x in X such that $A_{x,n} = \{\alpha \in A_n : P_\alpha \cap U_{x,n} \neq \emptyset\}$ is countable for every $n \in \mathbb{N}$. This implies that $f^{-1}(U_{x,n}) \subset \prod_{n\in\mathbb{N}} A_{x,n}$, then $p_n(f^{-1}(U_{x,n})) \subset A_{x,n}$. Since $A_{x,n}$ is countable, $p_n(f^{-1}(U_{x,n}))$ is separable. Then f is an $msss$-mapping .

(3). *Necessity.* Let f be an $mssc$-mapping. If there exists $k \in \mathbb{N}$ such that $\mathcal{P}_k$ is not locally finite, then, by using notations and arguments in the necessity of (2) again, we have that $A_{x,k}$ is infinite and $A_{x,k} \subset p_k(f^{-1}(U_x))$. Therefore, $\overline{p_k(f^{-1}(U_x))}$ is not compact. This is a contradiction to the fact that f is an $mssc$-mapping.

Sufficiency. Let each $\mathcal{P}_n$ be locally finite. By using notations and arguments in the sufficiency of (2) again, we have that $A_{x,n}$ is finite and $p_n(f^{-1}(U_{x,n})) \subset A_{x,n}$ for every $n \in \mathbb{N}$. Then $\overline{p_n(f^{-1}(U_{x,n}))}$ is compact. This implies that f is an $mssc$-mapping. □

In [9], [10], [11], [12], necessary and sufficient conditions such that f is a covering-mapping have been obtained in Ponomarev-systems $(f, M, X, \mathcal{P})$ and $(f, M, X, \{\mathcal{P}_n\})$. Next, we state necessary and sufficient conditions such that f is a covering-mapping in a σ-Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$.

Theorem 3.2. *Let $(f, M, X, \{\mathcal{P}_n\})$ be a σ-Ponomarev-system. Then the following hold.*

(1) *f is sequence-covering if and only if $\mathcal{P}$ is a strong cs-network for X.*
(2) *f is compact-covering if and only if $\mathcal{P}$ is a strong cfp-network for X.*
(3) *f is pseudo-sequence-covering if and only if $\mathcal{P}$ is a strong wcs-network for X.*
(4) *f is sequentially-quotient (subsequence-covering) if and only if $\mathcal{P}$ is a strong cs*-network for X.*

Proof. (1). *Necessity.* Let f be a sequence-covering mapping. Then for each convergent sequence S in X, there exists a convergent sequence C in M such that $f(C) = S$. Put $B = \bigcup\{p_n(C) : n \in \mathbb{N}\}$, and let $\mathcal{P}_S$ be the family of all finite intersections of members of $\{P_\alpha : \alpha \in B\}$. Then $\mathcal{P}_S$ is countable. Since $\mathcal{P}$ is closed under finite intersections, $\mathcal{P}_S \subset \mathcal{P}$. We shall prove that $\mathcal{P}_S$ is a cs-network for S in X. Let L be a convergent sequence in S converging to $x \in U$ with U open in X. Then there exists a convergent sequence $T \subset C$ such that $f(T) = L$. We have that T converges to some $a \in f^{-1}(x) \subset f^{-1}(U)$. Let $a = (\alpha_n)$, for each $n \in \mathbb{N}$ put

$$U_n = \{b = (\beta_i) \in M : \beta_i = \alpha_i \text{ if } i \leq n\}.$$

It follows from Lemma 2.11 that $\{U_n : n \in \mathbb{N}\}$ is a base at a in M. Since T converges to $a \in f^{-1}(U)$ which is open in M, T is eventually in some $U_n \subset f^{-1}(U)$. Therefore, L is eventually in $f(U_n) \subset U$. Since $f(U_n) = \bigcap_{i=1}^n P_{\alpha_i}$ by Lemma 2.11 and $\bigcap_{i=1}^n P_{\alpha_i} \in \mathcal{P}_S$, we get that $\mathcal{P}_S$ is a cs-network for S in X.

Sufficiency. Let $\mathcal{P}$ be a strong cs-network for X. For each sequence $S = \{x_m : m \in \omega\}$ converging to x_0 in X, (assume that all x_m's are distinct, if necessary), there exists $\mathcal{P}_S \subset \mathcal{P}$ such that $\mathcal{P}_S$ is a countable cs-network for S in X. We have that $\mathcal{F} = \{X\} \subset \mathcal{P}_S \cup \{X\}$ has property $cs(S, X)$. Since $\mathcal{P}_S$ is countable, $\{\mathcal{F} \subset \mathcal{P}_S \cup \{X\} : \mathcal{F} \text{ has property } cs(S, X)\}$ is countable. So we can put

$$\big\{\mathcal{F} \subset \mathcal{P}_S \cup \{X\} : \mathcal{F} \text{ has property } cs(S, X)\big\} = \{\mathcal{F}_i : i \in \mathbb{N}\},$$

and put $\mathcal{F}_{n(1)} = \{X\} \subset \mathcal{P}_1 \cap (\mathcal{P}_S \cup \{X\})$. For each $i \geq 2$, if there exists $j \in \mathbb{N}$ such that $\mathcal{F}_j \subset \big(\mathcal{P}_i \cap (\mathcal{P}_S \cup \{X\}) - \{\mathcal{F}_{n(k)} : k = 1, \dots, i-1\}\big)$, then put

$$n(i) = \min\big\{j \in \mathbb{N} : \mathcal{F}_j \subset \big(\mathcal{P}_i \cap (\mathcal{P}_S \cup \{X\}) - \{\mathcal{F}_{n(k)} : k = 1, \dots, i-1\}\big)\big\};$$

otherwise, put $\mathcal{F}_{n(i)} = \{X\}$. Then $\{\mathcal{F}_{n(i)} : i \in \mathbb{N}\} = \{\mathcal{F}_i : i \in \mathbb{N}\}$. Put $\mathcal{F}_{n(i)} = \{P_\alpha : \alpha \in B_i\}$, where $B_i \subset A_i$ is finite. For every $m \in \omega$ and $i \in \mathbb{N}$, since $\mathcal{F}_{n(i)}$ has property $cs(S, X)$, there exists a unique $\alpha_{im} \in B_i$ such that $x_m \in P_{\alpha_{im}} \in \mathcal{F}_{n(i)}$. Put $a_m = (\alpha_{im}) \in \prod_{i\in\mathbb{N}} B_i$ and $C = \{a_m : m \in \omega\}$, we shall prove that C is a convergent sequence in M and $f(C) = S$.

To show $C \subset M$ and $f(C) = S$ it suffices to prove that $\{P_{\alpha_{im}} : i \in \mathbb{N}\}$ is a network in X at x_m for every $m \in \omega$. Indeed, let U be an open neighborhood of x_m in X. We consider two following cases (a) and (b).

(a) $x_m = x_0$.

We have that $U \cap S$ is a convergent sequence in X and $S \cap U \subset U$. It follows from Lemma 2.6 that there exists a subfamily $\mathcal{F}$ of $\mathcal{P}_S$ such that $\mathcal{F}$ has property $cs(S \cap U, U)$. Since $S - (S \cap U)$ is finite, put $S - (S \cap U) = \{x_{m_i} : i = 1, \dots, l\}$ for some

$l \in \mathbb{N}$. For each $i \in \{1, \ldots, l\}$, note that $X - (S - \{x_{m_i}\})$ is an open neighborhood of x_{m_i} in X, so there exists $P_i \in \mathcal{P}_S$ such that $x_{m_i} \in P_i \subset X - (S - \{x_{m_i}\})$. It is easy to see that $\mathcal{F} \cup \{P_i : i = 1, \ldots, l\}$ has property $cs(S, X)$. So there exists $k \in \mathbb{N}$ such that $\mathcal{F} \cup \{P_i : i = 1, \ldots, l\} = \mathcal{F}_{n(k)}$. Thus $x_0 \in P_{\alpha_{k0}} \in \mathcal{F}_{n(k)}$. Because $P_{\alpha_{k0}}$ must be an element of $\mathcal{F}$ which has property $cs(S \cap U, U)$, $x_0 \in P_{\alpha_{k0}} \subset U$.

(b) $x_m \neq x_0$.

We have that $S - \{x_m\}$ is a convergent sequence in X and $S - \{x_m\} \subset X - \{x_m\}$ with $X - \{x_m\}$ open. It follows from Lemma 2.6 that there exists a subfamily $\mathcal{F}$ of $\mathcal{P}_S$ such that $\mathcal{F}$ has property $cs(S - \{x_m\}, X - \{x_m\})$. Note that $U - (S - \{x_m\})$ is an open neighborhood of x_m, so there exists $P_m \in \mathcal{P}_S$ such that $x_m \in P_m \subset U - (S - \{x_m\})$. It is easy to see that $\mathcal{F} \cup \{P_m\}$ has property $cs(S, X)$. Hence there exists $k \in \mathbb{N}$ such that $\mathcal{F} \cup \{P_m\} = \mathcal{F}_{n(k)}$, then $x_m \in P_{\alpha_{km}} = P_m \subset U$.

By the above cases, there exists $k \in \mathbb{N}$ such that $x_m \in P_{\alpha_{km}} \subset U$ for every $m \in \omega$. Then $\{P_{\alpha_{im}} : i \in \mathbb{N}\}$ is a network in X at x_m for every $m \in \omega$. This implies that $C \subset M$ and $f(C) = S$. To complete the proof we shall prove that $C = \{a_m : m \in \omega\}$ converges to a_0. For every $k \in \mathbb{N}$ there exists a unique $\alpha_{k0} \in B_k$ such that $x_0 \in P_{\alpha_{k0}} \in \mathcal{F}_{n(k)}$. Since $\mathcal{F}_{n(k)}$ has property $cs(S, X)$, $S - P_{\alpha_{k0}}$ is finite. So there exists $m_k \in \mathbb{N}$ such that $x_m \in P_{\alpha_{k0}}$ for every $m > m_k$. Note that $x_m \in P_{\alpha_{km}} \in \mathcal{F}_{n(k)}$. Thus $\alpha_{km} = \alpha_{k0}$ for every $m > m_k$. So $C = \{a_m : m \in \omega\}$ converges to a_0 in M. This implies that $S = f(C)$ with C being a convergent sequence in M, hence f is a sequence-covering mapping.

(2). *Necessity.* Let f be a compact-covering mapping. Then for each compact subset K of X, there exists a compact subset C of M such that $f(C) = K$. Put $B = \bigcup\{p_n(C) : n \in \mathbb{N}\}$, and let $\mathcal{P}_K$ be the family of all finite intersections of members of $\{P_\alpha : \alpha \in B\}$. Then $\mathcal{P}_K$ is countable. Since $\mathcal{P}$ is closed under finite intersections, $\mathcal{P}_K \subset \mathcal{P}$. We shall prove that $\mathcal{P}_K$ is a strong cfp-network for K in X. Let H be a compact subset of K and $H \subset U$ with U open in X. Then $L = f^{-1}(H) \cap C$ is compact. For each $a = (\alpha_n) \in L$, we have that $\alpha_n \in A_n$ for every $n \in \mathbb{N}$, and $\{P_{\alpha_n} : n \in \mathbb{N}\}$ is a network at some point $x_a = f(a) \in H$ in X. Then there exists $k \in \mathbb{N}$ such that $x_a \in P_{\alpha_k} \subset U$. Put $U_{a,k} = \{b = (\beta_n) \in M : \beta_n = \alpha_n \text{ if } n \leq k\}$. Then $U_{a,k} \cap L$ is an open neighborhood of a in L. So there exists an open neighborhood $V_{a,k}$ of a in L such that $a \in V_{a,k} \subset \overline{V}_{a,k} \subset U_{a,k} \cap L$. Since $\{V_{a,k} : a \in L\}$ is an open cover for L and L is compact, there exists $\{a_1, \ldots, a_m\} \subset L$ such that $\{V_{a_1,k}, \ldots, V_{a_m,k}\}$ is a finite cover for L. It is easy to see that $\bigcup\{\overline{V}_{a_i,k} : i = 1, \ldots, m\} = L$, and so $\bigcup\{f(\overline{V}_{a_i,k}) : i = 1, \ldots, m\} = f(\bigcup\{\overline{V}_{a_i,k} : i = 1, \ldots, m\}) = f(L) = H$. For each $i \in \{1, \ldots, m\}$, put $H_i = f(\overline{V}_{a_i,k})$ and $a_i = (\alpha_{in})_n$. Then each H_i is closed, and $H = \bigcup\{H_i : i = 1, \ldots, m\}$. On the other hand, $f(U_{a_i,k}) \subset P_{\alpha_{ik}}$ by Lemma 2.11, then $H_i \subset f(U_{a_i,k} \cap L) \subset f(U_{a_i,k}) \subset P_{\alpha_{ik}}$. This proves that $\mathcal{P}_K$ is a cfp-network for K in X.

Sufficiency. Let $\mathcal{P}$ be a strong cfp-network for X. For each compact subset K of X, there exists $\mathcal{P}_K \subset \mathcal{P}$ such that $\mathcal{P}_K$ is a countable cfp-network for K in X. We have that $\mathcal{F} = \{X\} \subset \mathcal{P}_K \cup \{X\}$ has property $cfp(K, X)$. Since $\mathcal{P}_K$ is countable, $\{\mathcal{F} \subset \mathcal{P}_K \cup \{X\} : \mathcal{F}$ has property $cfp(K, X)\}$ is countable. So we can put

$$\big\{\mathcal{F} \subset \mathcal{P}_K \cup \{X\} : \mathcal{F} \text{ has property } cfp(K, X)\big\} = \{\mathcal{F}_i : i \in \mathbb{N}\},$$

and put $\mathcal{F}_{n(1)} = \{X\} \subset \mathcal{P}_1 \cap (\mathcal{P}_K \cup \{X\})$. For each $i \geq 2$, if there exists $j \in \mathbb{N}$ such that $\mathcal{F}_j \subset \big(\mathcal{P}_j \cap (\mathcal{P}_K \cup \{X\}) - \{\mathcal{F}_{n(k)} : k = 1, \ldots, i-1\}\big)$, then put

$$n(i) = \min\big\{j \in \mathbb{N} : \mathcal{F}_j \subset \big(\mathcal{P}_j \cap (\mathcal{P}_K \cup \{X\}) - \{\mathcal{F}_{n(k)} : k = 1, \ldots, i-1\}\big)\big\};$$

otherwise, put $\mathcal{F}_{n(i)} = \{X\}$. Then $\{\mathcal{F}_{n(i)} : i \in \mathbb{N}\} = \{\mathcal{F}_i : i \in \mathbb{N}\}$. Put $\mathcal{F}_{n(i)} = \{P_\alpha : \alpha \in B_i\}$, where B_i is a finite set, and put

$$C = \Big\{a = (\alpha_n) \in \prod_{n \in \mathbb{N}} B_n : \bigcap_{n \in \mathbb{N}} (P_{\alpha_n} \cap K) \neq \emptyset\Big\}.$$

We shall prove that C is a compact subset of M and $f(C) = K$ by the following facts (a), (b), and (c).

(a) $C \subset M$ and $f(C) \subset K$.

Let $a = (\alpha_n) \in C$, then $\bigcap_{n \in \mathbb{N}}(P_{\alpha_n} \cap K) \neq \emptyset$. Pick $x \in \bigcap_{n \in \mathbb{N}}(P_{\alpha_n} \cap K)$. Then it suffices to show that $\{P_{\alpha_n} : n \in \mathbb{N}\}$ is a network at x in X. In this case, $a \in M$ and $f(a) = x \in K$, so $C \subset M$ and $f(C) \subset K$.

Let V be a neighborhood of x in X. Then there exist open neighborhoods W_1, W_2 of x in K such that $x \in W_1 \subset \overline{W_1} \subset W_2 \subset \overline{W_2} \subset V \cap K$. Since $\overline{W_2}$ is a compact subset of K, there exists $\mathcal{R}_1 \subset \mathcal{P}_K$ such that $\mathcal{R}_1$ has property $cfp(\overline{W_2}, V)$. On the other hand, $K - W_2$ is also a compact subset of K and $K - W_2 \subset X - \overline{W_1}$, so there exists $\mathcal{R}_2 \subset \mathcal{P}_K$ such that $\mathcal{R}_2$ has property $cfp(K - W_2, X - \overline{W_1})$. Then there exists $\mathcal{F} \subset \mathcal{R}_1 \cup \mathcal{R}_2$ such that $\mathcal{F}$ has property $cfp(K, X)$. This implies that $\mathcal{F} = \mathcal{F}_{n(i)}$ for some $i \in \mathbb{N}$, and then $x \in P_{\alpha_i} \in \mathcal{F}$. By our construction, $P_{\alpha_i} \in \mathcal{R}_1$. Then $x \in P_{\alpha_i} \subset V$, hence $\{P_{\alpha_n} : n \in \mathbb{N}\}$ is a network at x in X.

(b) $K \subset f(C)$.

For each $x \in K$ and each $i \in \mathbb{N}$, pick $P_{\alpha_i} \in \mathcal{F}_{n(i)}$ such that $x \in P_{\alpha_i}$. Then $\bigcap_{i \in \mathbb{N}}(P_{\alpha_i} \cap K) \neq \emptyset$. This implies that $a = (\alpha_i) \in C$, and $\{P_{\alpha_i} : i \in \mathbb{N}\}$ forms a network at x in X as in the proof of (a). Then $f(a) = x$. It shows that $x \in f(C)$, i.e., $K \subset f(C)$.

(c) C is a compact subset of M.

Because $C \subset \prod_{n \in \mathbb{N}} B_n$ and $\prod_{n \in \mathbb{N}} B_n$ is a compact subset of $\prod_{n \in \mathbb{N}} A_n$, it suffices to prove that C is closed in $\prod_{n \in \mathbb{N}} B_n$. Let $a = (\alpha_n) \in \prod_{n \in \mathbb{N}} B_n - C$, then $\bigcap_{n \in \mathbb{N}}(P_{\alpha_n} \cap K) = \emptyset$. Since each $P_{\alpha_n} \cap K$ is compact, there exists $k \in \mathbb{N}$ such that $\bigcap_{n \leq k}(P_{\alpha_n} \cap K) = \emptyset$. Put $U = \{b = (\beta_n) \in \prod_{n \in \mathbb{N}} B_n : \beta_n = \alpha_n \text{ if } n \leq k\}$. Then U is an open neighborhood of a in $\prod_{n \in \mathbb{N}} B_n$ and $U \cap C = \emptyset$. This implies that C is closed in $\prod_{n \in \mathbb{N}} B_n$.

(3). By Remark 2.2 and using arguments as in (2), where "*compact-covering*", "*a compact subset*", and "*strong cfp-network*" are replaced by "*pseudo-sequence-covering*", "*a convergent sequence*", and "*strong wcs-network*" respectively.

(4). *Necessity.* Let f be a sequentially-quotient mapping. Then for each convergent sequence S in X, there exists a convergent sequence C in M such that $f(C)$ is a subsequence of S. By using arguments as in the necessity of (1) again, we have that there exists $\mathcal{P}_{f(C)} \subset \mathcal{P}$ such that $\mathcal{P}_{f(C)}$ is a countable *cs*-network for $f(C)$ in X. This implies that $\mathcal{P}$ is a strong cs^*-network for X. For the parenthetic part, it is obvious by the fact that each subsequence-covering mapping is a sequentially-quotient mapping.

Sufficiency. Let $\mathcal{P}$ be a strong cs^*-network for X. Then for each convergent sequence S in X, there exists a countable $\mathcal{P}_K$ such that $\mathcal{P}_K$ is a cs^*-network for some subsequence K of S in X. By Remark 2.2 and using arguments as in the sufficiency of (2), we have that there exists a compact subset C of M such that $f(C) = K$. This implies that f is subsequence-covering, then f is sequentially-quotient by [7, Proposition 2.1]. □

By using Theorem 3.1 and Theorem 3.2, we systematically get characterizations of images of metric spaces under s-mappings ($msss$-mappings, $mssc$-mappings) with covering-properties as in [8], [16], [17], [20], and others as follows.

Corollary 3.3 ([8], Theorem 5)**.** *The following are equivalent for a space X.*

(1) *X is an $\aleph$-space.*
(2) *X is a sequence-covering mssc-image of a metric space.*
(3) *X is a pseudo-sequence-covering mssc-image of a metric space.*
(4) *X is a subsequence-covering mssc-image of a metric space.*
(5) *X is a sequentially-quotient mssc-image of a metric space.*

Proof. The main proof is (1) $\Rightarrow$ (2), other implications are easy. Let $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a σ-locally finite *cs*-network for X. Then the σ-Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$ exists. Since $\mathcal{P}$ is σ-locally finite, $\mathcal{P}$ is a strong *cs*-network for X. It follows from Theorem 3.1 and Theorem 3.2 that f is a sequence-covering $mssc$-mapping from a metric space M onto X. □

By using arguments as in the proof of Corollary 3.3, we get the following results, which partly appeared in [17], [18], and [20].

Corollary 3.4. *The following are equivalent for a space X. We can replace "σ-locally countable" and "msss-image" by "point-countable" and "s-image", and replace "cs-network" and "sequence-covering" by "cs^*-network" and "sequentially-quotient"("wcs-network" and "pseudo-sequence-covering", "cfp-network" and "compact-covering") respectively.*

(1) *X has a σ-locally countable cs-network.*
(2) *X is a sequence-covering msss-image of a metric space.*

Corollary 3.5. *The following are equivalent for a space X.*

(1) *X has a point-countable cs^*-network.*
(2) *X is a pseudo-sequence-covering s-image of a metric space.*
(3) *X is a subsequence-covering s-image of a metric space.*

(4) *X is a sequentially-quotient s-image of a metric space.*

Related to characterizations of images of metric spaces, many authors were interested in that of separable metric spaces. In [24], E. Michael characterized compact-covering images (resp., images) of separable metric spaces by $\aleph_0$-spaces (resp., cosmic spaces). Recently, some nice results on spaces with countable networks have been obtained. In [29, Corollary 8], Y. Tanaka and Z. Li proved that a space X has a countable cs^*-network (resp., cs-network) if and only if X is a pseudo-sequence-covering (resp., sequence-covering) image of a separable metric space. Next, based on the σ-Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$, we get new results on spaces having certain countable network as follows.

Corollary 3.6. *The following are equivalent for a space X.*

(1) *X is a cosmic space.*
(2) *X is a rc-mssc-image of a separable metric space.*
(3) *X is an mssc-image of a separable metric space.*
(4) *X is an image of a separable metric space.*

Proof. We only need to prove (1) $\Rightarrow$ (2), other implications are easy. Since X is a cosmic space, X has a countable network $\mathcal{P} = \{P_i : i \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, put $\mathcal{P}_n = \{X\} \cup \{P_i : i \leq n\}$. Then $\mathcal{P} \cup \{X\} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a σ-locally finite network for X, so the σ-Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$ exists. Since each $\mathcal{P}_n$ is finite, $\prod_{n\in\mathbb{N}} A_n$ is a compact space. This implies that M is relatively compact in $\prod_{n\in\mathbb{N}} A_n$. Therefore, f is a *rc-mssc*-mapping by Theorem 3.1. □

Corollary 3.7. *The following are equivalent for a space X.*

(1) *X is an $\aleph_0$-space.*
(2) *X is a sequence-covering, compact-covering rc-mssc-image of a separable metric space.*
(3) *X is a sequence-covering, compact-covering mssc-image of a separable metric space.*
(4) *X is a sequentially-quotient image of a separable metric space.*

Proof. We only need to prove (1) $\Rightarrow$ (2), other implications are easy. Since X is an $\aleph_0$-space, X has a countable cs-network $\mathcal{Q}$ and a countable cfp-network $\mathcal{R}$. Put $\mathcal{P} = \mathcal{Q} \cup \mathcal{R}$, then $\mathcal{P}$ is a strong cs-network and cfp-network for X. Put $\mathcal{P} = \{P_i : i \in \mathbb{N}\}$, and put $\mathcal{P}_n = \{P_i : i \leq n\} \cup \{X\}$. Then $\mathcal{P} \cup \{X\} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a σ-locally finite cs-network and cfp-network for X, so the σ-Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$ exists. Since each $\mathcal{P}_n$ is finite, $\prod_{n\in\mathbb{N}} A_n$ is a compact space. Then M is relatively compact in $\prod_{n\in\mathbb{N}} A_n$. It follows from Theorem 3.1 and Theorem 3.2 that f is a sequence-covering, compact-covering *rc-mssc*-mapping from a separable metric space M onto X. □

The following example shows that Corollary 3.6 and Corollary 3.7 are better results than preceding ones of E. Michael [24], Y.Tanaka and Z. Li [29].

Example 3.8. A sequence-covering, compact-covering mapping from a separable metric space is not an *mssc*-mapping.

Proof. Recall that $\mathbb{Q} \subset \mathbb{R}$ is a non-locally compact, separable metric space, where $\mathbb{Q}$ is the set of all rational numbers and $\mathbb{R}$ is the set of all real numbers with the usual topology. Put $M = \mathbb{Q} \times \{0\} \times \cdots \times \{0\} \cdots \subset \prod_{i\in\mathbb{N}} X_i$, where $X_i = \mathbb{Q}$ for every $i \in \mathbb{N}$. Then M is a separable metric space. Define $f : M \longrightarrow \mathbb{Q}$ by $f(x, 0, \dots) = x$ for each $x \in \mathbb{Q}$. Then f is a sequence-covering, compact-covering mapping from a separable metric space onto $\mathbb{Q}$. If f is an *mssc*-mapping, then, for every $x \in \mathbb{Q}$, there exists a sequence $\{V_{x,i} : i \in \mathbb{N}\}$ of open neighborhoods of x in $\mathbb{Q}$ such that each $\overline{p_i(f^{-1}(V_{x,i}))}$ is a compact subspace of X_i. Thus, $\overline{p_1(f^{-1}(V_{x,1}))}$ is a compact subset of $\mathbb{Q}$, so $\mathbb{Q}$ is a locally compact space. This is a contradiction. Hence f is not an *mssc*-mapping. □

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