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ON SOME GENERALIZED NEW TYPE DIFFERENCE SEQUENCE SPACES DEFINED BY A MODULUS FUNCTION

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Abstract. The idea of difference sequence spaces were defined by Kizmaz [6] and generalized by Et and Çolak [3]. Later Tripathy et al. [16] introduced the notion of the new difference operator $\Delta_m^n x_k$ for fixed $n, m \in \mathbb{N}$. In this paper we introduce some new type difference sequence spaces defined by a modulus function and the new concept of statistical convergence. We give various properties and inclusion relations on these new type difference sequence spaces.

1. Introduction

The difference sequence spaces $X(\Delta)$ was introduced by Kizmaz [6] as follows:

$$X(\Delta) = \{x = (x_k) : \ (\Delta x_k) \in X\} \text{ for } X = l_\infty, c \text{ and } c_0,$$

where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$. Later, the difference sequence spaces were generalized by Et and Çolak [3] as follows: Let $n \in \mathbb{N}$ be fixed, then

$$X(\Delta^n) = \{x = (x_k) : \ (\Delta^n x_k) \in X\} \text{ for } X = l_\infty, c \text{ and } c_0,$$

where $\Delta^n x_k = \Delta^{n-1} x_k - \Delta^{n-1} x_{k+1}$ and so $\Delta^n x_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k+v}$. Quite recently, this operator was generalized by Tripathy et al. as follows: Let $n, m \in \mathbb{N}$ be fixed, then

$$X(\Delta_m^n) = \{x = (x_k) : \ (\Delta_m^n x_k) \in X\} \text{ for } X = l_\infty, c \text{ and } c_0,$$

where $\Delta_m^n x_k = \Delta_m^{n-1} x_k - \Delta_m^{n-1} x_{k+1}$ and $\Delta_m^0 x_k = x_k$ for all $k \in \mathbb{N}$. This generalized notion has the following binomial representation:

$$\Delta_m^n x_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k+mv}.$$

The notion of modulus function was introduced by Nakano [13] and Ruckle [15]. We recall that a modulus f is a function from $[0,\infty)$ to $[0,\infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$,

(ii) $f(x+y) \le f(x) + f(y)$,

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(iii) f is increasing, and

(iv) f is continuous from the right at 0.

It is immediate from (ii) and (iv) that f is continuous on $[0,\infty)$. Also, from condition (ii), we have $f(nx) \leq nf(x)$ for all $n \in \mathbb{N}$, and so $f(x) \leq f\left(nx\frac{1}{n}\right) \leq nf\left(\frac{x}{n}\right)$. Hence $\frac{1}{n}f(x) \leq f\left(\frac{x}{n}\right)$ for all $n \in \mathbb{N}$. A modulus function may be bounded (for example, $f(x) = \frac{x}{1+x}$) or unbounded (for example, $f(x) = x$). Ruckle [15], Maddox [11], Esi [2] and several authors used a modulus f to construct some sequence spaces.

Spaces of strongly summable sequences were discussed by Kuttner [7], Maddox [9] and others. The class of sequences which are strongly Cesaro summable with recpect to a modulus was introduced by Maddox [11] as an extension of the definition of strongly Cesaro summable sequences. Connor [1] further extended this definition to a definition of strongly $A-$*summability* with recpect to a modulus when A is non-negative regular matrix.

Let $\Lambda = (\lambda_i)$ be a non-decreasing sequence of positive real numbers tending to infinity and $\lambda_1 = 1$ and $\lambda_{i+1} \leq \lambda_i + 1$.

The generalized de la Vallee-Poussin means is defined by

$$t_i(x) = \frac{1}{\lambda_i}\sum_{k\in I_i} x_k,$$

where $I_i = [i-\lambda_i+1, i]$. A sequence $x=(x_k)$ is said to be (V,λ)-summable to a number L if $t_i(x) \to L$ as $i \to \infty$ (see [8]). We write

$$[V,\lambda]_0 = \left\{x=(x_k): \ \lim_i \frac{1}{\lambda_i}\sum_{k\in I_i}|x_k| = 0\right\},$$

$$[V,\lambda] = \left\{x=(x_k): \ \lim_i \frac{1}{\lambda_i}\sum_{k\in I_i}|x_k - L| = 0, \text{ for some } L\right\},$$

$$\text{and } [V,\lambda]_\infty = \left\{x=(x_k): \sup_i \frac{1}{\lambda_i}\sum_{k\in I_i}|x_k| < \infty\right\}.$$

For the sets of sequences that are called $\lambda-$*strongly* summable to zero, $\lambda-$*strongly* summable and $\lambda-$*strongly* bounded by de la Vallee-Poussin method. In the special case, where $\lambda_i = 1$ for all $i = 1,2,3,...$ the sets $[V,\lambda]_0, [V,\lambda]$ and $[V,\lambda]_\infty$ reduce to the sets w_0, w and w_∞ introduced and studied by Maddox [9].

The following inequality will be used throughout this paper:

$$|a_k + b_k|^{p_k} \leq C\left(|a_k|^{p_k} + |b_k|^{p_k}\right), \tag{1.1}$$

where a_k and b_k are complex numbers, $C = \max\left(1, 2^{H-1}\right)$, $H = \sup_k p_k < \infty$ [16].

2. Main results

Definition 2.1. Let E be a Banach space. We define $w(E)$ to be vector space of all E-*valued* sequences, that is

$$w(E) = \{x = (x_k) : \quad x_k \in E\}.$$

Let f be a modulus function, $p = (p_k)$ be any sequence of strictly positive real numbers, $A = (a_{sk})$ be a non-negative matrix such that $\sup_s \sum_{k=1}^{\infty} a_{sk} < \infty$ and $n, m \in \mathbb{N}$ be fixed. (This assumption is made throughout the rest of this paper). We define the following sets:

$$\left[V_\lambda^E, A, \Delta_m^n, f, p\right]_0 = \left\{ x = (x_k) \in w(E) : \lim_i \frac{1}{\lambda_i} \sum_{k \in I_i} a_{sk} \left[f\left(\left\|\Delta_m^n x_k\right\|\right)\right]^{p_k} = 0 \right\},$$

$$\left[V_\lambda^E, A, \Delta_m^n, f, p\right]_1 = \left\{ x = (x_k) \in w(E) : \lim_i \frac{1}{\lambda_i} \sum_{k \in I_i} a_{sk} [f(\|\Delta_m^n x_k - L\|)]^{p_k} = 0, \text{ for some } L \right\}, \text{ and}$$

$$\left[V_\lambda^E, A, \Delta_m^n, f, p\right]_\infty = \left\{ x = (x_k) \in w(E) : \sup_i \frac{1}{\lambda_i} \sum_{k \in I_i} a_{sk} \left[f\left(\left\|\Delta_m^n x_k\right\|\right)\right]^{p_k} < \infty \right\}.$$

If $x \in \left[V_\lambda^E, A, \Delta_m^n, f, p\right]_1$ then we write $x \to L\left(\left[V_\lambda^E, A, \Delta_m^n, f, p\right]_1\right)$ and L will be called $\lambda_E^{n,m}$−new type difference limit of $x = (x_k)$ with recpect to the modulus function f.

If $a_{sk} = 1$ for all $s, k \in \mathbb{N}$ and $m = 0$, then $\left[V_\lambda^E, A, \Delta_m^n, f, p\right]_0$, $\left[V_\lambda^E, A, \Delta_m^n, f, p\right]_1$ and $\left[V_\lambda^E, A, \Delta_m^n, f, p\right]_\infty$ reduce to $[V, \lambda, p]_0 (\Delta^n, E)$, $[V, \lambda, p]_1 (\Delta^n, E)$ and $[V, \lambda, p]_\infty$ (Δ^n, E) which were studied by Et et al. [11].

Throughout the paper Z will denote any one of the notation 0, 1 or ∞.

Proposition 2.1. *Let the sequence $p = (p_k)$ be bounded. Then the sequence spaces $\left[V_\lambda^E, A, \Delta_m^n, f, p\right]_Z$ are linear spaces over the complex field $\mathbb{C}$ for $Z = 0,\ 1,$ or ∞.*

Proof. We consider only $\left[V_\lambda^E, A, \Delta_m^n, f, p\right]_0$. The others can be treated similarly. Let $x, y \in \left[V_\lambda^E, A, \Delta_m^n, f, p\right]_0$ and $\gamma, \mu \in \mathbb{C}$. Then there exist positive numbers M_γ and N_μ such that $|\gamma| \le M_\gamma$ and $|\mu| \le N_\mu$. Since f is subadditive and the operation Δ_m^n is linear

$$\begin{aligned} &\frac{1}{\lambda_i} \sum_{k \in I_i} a_{sk} \left[f\left(\left\|\Delta_m^n \left(\gamma x_k + \mu y_k\right)\right\|\right)\right]^{p_k} \\ \le &\frac{1}{\lambda_i} \sum_{k \in I_i} a_{sk} \left[f\left(|\gamma| \left\|\Delta_m^n x_k\right\|\right) + f\left(|\mu| \left\|\Delta_m^n y_k\right\|\right)\right]^{p_k} \end{aligned}$$

$$\leq C\left(M_{\gamma}\right)^{H}\frac{1}{\lambda_i}\sum_{k\in I_i}a_{sk}\left[f\left(\left\|\Delta_m^n x_k\right\|\right)\right]^{p_k}$$
$$+\,C\left(N_{\mu}\right)^{H}\frac{1}{\lambda_i}\sum_{k\in I_i}a_{sk}\left[f\left(\left\|\Delta_m^n y_k\right\|\right)\right]^{p_k}\to 0 \text{ as } n\to\infty.$$

This proves that $\left[V_{\lambda}^{E},A,\Delta_m^n,f,p\right]_0$ is a linear space. □

Proposition 2.2. *Let f be a modulus, then*

$$\left[V_{\lambda}^{E},A,\Delta_m^n,f,p\right]_0\subset\left[V_{\lambda}^{E},A,\Delta_m^n,f,p\right]_1\subset\left[V_{\lambda}^{E},A,\Delta_m^n,f,p\right]_{\infty}.$$

Proof. The first inclusion is obvious. We establish the second inclusion. Let $x\in\left[V_{\lambda}^{E},A,\Delta_m^n,f,p\right]_1$. By definition of modulus f, we have

$$\frac{1}{\lambda_i}\sum_{k\in I_i}a_{sk}\left[f\left(\left\|\Delta_m^n x_k\right\|\right)\right]^{p_k}$$
$$\leq\frac{C}{\lambda_i}\sum_{k\in I_i}a_{sk}\left[f\left(\left\|\Delta_m^n x_k-L\right\|\right)\right]^{p_k}+\frac{C}{\lambda_i}\sum_{k\in I_i}a_{sk}\left[f\left(\left\|L\right\|\right)\right]^{p_k}.$$

There exists a positive integer M_L such that $\|L\|\leq M_L$. Hence we have

$$\frac{1}{\lambda_i}\sum_{k\in I_i}a_{sk}\left[f\left(\left\|\Delta_m^n x_k\right\|\right)\right]^{p_k}$$
$$\leq\frac{C}{\lambda_i}\sum_{k\in I_i}a_{sk}\left[f\left(\left\|\Delta_m^n x_k-L\right\|\right)\right]^{p_k}+\frac{C\left(M_L f\left(1\right)\right)^{H}}{\lambda_i}\sum_{k\in I_i}a_{sk}.$$

Since $x\in\left[V_{\lambda}^{E},A,\Delta_m^n,f,p\right]_1$, we have $x\in\left[V_{\lambda}^{E},A,\Delta_m^n,f,p\right]_{\infty}$ and this completes the proof. □

Theorem 2.3. *The sequence space $\left[V_{\lambda}^{E},A,\Delta_m^n,f,p\right]_0$ is a paranormed space with*

$$g_{\Delta_m^n}\left(x\right)=\sup_i\left(\frac{1}{\lambda_i}\sum_{k\in I_i}a_{sk}\left[f\left(\left\|\Delta_m^n x_k\right\|\right)\right]^{p_k}\right)^{\frac{1}{M}},$$

where $M=\max\left(1,\sup_k p_k\right)$.

Proof. From Proposition 2.2, for each $x\in\left[V_{\lambda}^{E},A,\Delta_m^n,f,p\right]_0$, $g_{\Delta_m^n}(x)$ exists. Clearly, $g_{\Delta_m^n}(x)=g_{\Delta_m^n}(-x)$. It is trivial that $\Delta_m^n x_k=0$ for $x=0$. Since $f(0)=0$, we get $g_{\Delta_m^n}(x)=0$ for $x=0$ and by Minkowski's Inequality $g_{\Delta_m^n}(x+y)\leq g_{\Delta_m^n}(x)+g_{\Delta_m^n}(y)$. We now show that the scalar multiplication is continuous. Let γ be any complex number. By definition of modulus f, we have

$$g_{\Delta_m^n}\left(\gamma x\right)=\sup_i\left(\frac{1}{\lambda_i}\sum_{k\in I_i}a_{sk}\left[f\left(\left\|\Delta_m^n\gamma x_k\right\|\right)\right]^{p_k}\right)^{\frac{1}{M}}\leq N_{\gamma}^{\frac{H}{M}}g_{\Delta_m^n}\left(x\right),$$

where N_γ is a positive integer such that $|\gamma| \le N_\gamma$. Now, $\gamma \to 0$ for any fixed $x = (x_k)$ with $g_{\Delta_m^n}(x) \neq 0$. By definition of modulus f, for $|\gamma| < 1$, we have

$$\frac{1}{\lambda_i} \sum_{k \in I_i} a_{sk} \left[f\left(\|\Delta_m^n \gamma x_k\| \right) \right]^{p_k} < \varepsilon \text{ for } i > i_0(\varepsilon). \tag{2.1}$$

Also, for $1 \le i \le i_0$, taking γ small enough, since f is continuous, we have

$$\frac{1}{\lambda_i} \sum_{k \in I_i} a_{sk} \left[f\left(\|\Delta_m^n \gamma x_k\| \right) \right]^{p_k} < \varepsilon. \tag{2.2}$$

(2.1) and (2.2) together imply that $g_{\Delta_m^n}(\gamma x) \to 0$ as $\gamma \to 0$. This completes the proof □

Proposition 2.4. *If $n \ge 1$, then the inclusion*

$$\left[V_\lambda^E, A, \Delta_m^{n-1}, f, p \right]_Z \subset \left[V_\lambda^E, A, \Delta_m^n, f, p \right]_Z$$

is strict for $Z = 0, 1,$ or ∞. In general $\left[V_\lambda^E, A, \Delta_m^j, f, p \right]_Z \subset \left[V_\lambda^E, A, \Delta_m^n, f, p \right]_Z$ for $j = 1, 2, ..., n-1$ and the inclusion is strict for $j = 1, 2, ..., n-1$.

Proof. We give the proof for $Z = \infty$ only. The others can be proved in a similar way for $Z = 0$ and $Z = 1$. Let $x \in \left[V_\lambda^E, A, \Delta_m^{n-1}, f, p \right]_Z$. Then we have

$$\sup_i \frac{1}{\lambda_i} \sum_{k \in I_i} a_{sk} \left[f\left(\left\| \Delta_m^{n-1} x_k \right\| \right) \right]^{p_k} < \infty.$$

By definition of modulus f, we have

$$\begin{aligned}
&\frac{1}{\lambda_i} \sum_{k \in I_i} a_{sk} \left[f\left(\|\Delta_m^n x_k\| \right) \right]^{p_k} \\
\le &\frac{1}{\lambda_i} \sum_{k \in I_i} a_{sk} \left[f\left(\left\| \Delta_m^{n-1} x_k \right\| \right) + f\left(\left\| \Delta_m^{n-1} x_{k+1} \right\| \right) \right]^{p_k} \\
\le &\frac{C}{\lambda_i} \sum_{k \in I_i} a_{sk} \left[f\left(\left\| \Delta_m^{n-1} x_k \right\| \right) \right]^{p_k} + \frac{C}{\lambda_i} \sum_{k \in I_i} a_{sk} \left[f\left(\left\| \Delta_m^{n-1} x_{k+1} \right\| \right) \right]^{p_k} < \infty.
\end{aligned}$$

Thus, $\left[V_\lambda^E, A, \Delta_m^{n-1}, f, p \right]_Z \subset \left[V_\lambda^E, A, \Delta_m^n, f, p \right]_Z$ for $j = 1, 2, ..., n-1$. Now, proceeding in this way one will have $\left[V_\lambda^E, A, \Delta_m^j, f, p \right]_\infty \subset \left[V_\lambda^E, A, \Delta_m^n, f, p \right]_\infty$ for $j = 1, 2, ..., n-1$. Let $E = \mathbb{C}$, $\lambda_i = i$ for each $i \in \mathbb{N}$ and $a_{sk} = 1$ for each $s, k \in \mathbb{N}$. Then the sequence $x = (k^n)$ belongs to $\left[V_\lambda^{\mathbb{C}}, A, \Delta_m^n, f, p \right]_\infty$ but it does not belong to $\left[V_\lambda^{\mathbb{C}}, A, \Delta_m^{n-1}, f, p \right]_\infty$ for $f(x) = x$ and $m = 0$. Note that, $x = (k^n)$, then $\Delta^n x_k = (-1)^n n!$ and $\Delta^{n-1} x_k = (-1)^{n+1} n! \left(k + \left(\frac{n-1}{2} \right) \right)$ for all $k \in \mathbb{N}$ and $m = 0$. □

Proposition 2.5. *Let f be a modulus function.*
(a) *If $0 < \inf_k p_k \le p_k \le 1$ for all $k \in \mathbb{N}$, then*

$$\left[V_\lambda^E, A, \Delta_m^n, f, p \right]_1 \subset \left[V_\lambda^E, A, \Delta_m^n, f \right]_1.$$

(b) *If* $1 \leq p_k \leq \sup_k p_k < \infty$ *for all* $k \in \mathbb{N}$, *then*

$$\left[V_\lambda^E, A, \Delta_m^n, f\right]_1 \subset \left[V_\lambda^E, A, \Delta_m^n, f, p\right]_1.$$

(c) *Let* $0 < p_k \leq q_k$ *for all* $k \in \mathbb{N}$ *and* $\left(\frac{q_k}{p_k}\right)$ *be bounded, then*

$$\left[V_\lambda^E, A, \Delta_m^n, f, q\right]_1 \subset \left[V_\lambda^E, A, \Delta_m^n, f, p\right]_1.$$

Proof. (a) Let $x \in \left[V_\lambda^E, A, \Delta_m^n, f, p\right]_1$. Since $0 < \inf_k p_k \leq p_k \leq 1$ for all $k \in \mathbb{N}$, we get

$$\frac{1}{\lambda_i} \sum_{k \in I_i} a_{sk} f\left(\left\|\Delta_m^n x_k - L\right\|\right) \leq \frac{1}{\lambda_i} \sum_{k \in I_i} a_{sk} \left[f\left(\left\|\Delta_m^n x_k - L\right\|\right)\right]^{p_k}$$

and hence $x \in \left[V_\lambda^E, A, \Delta_m^n, f\right]_1$.

(b) Let $1 \leq p_k \leq \sup_k p_k < \infty$ for all $k \in \mathbb{N}$ and $x \in \left[V_\lambda^E, A, \Delta_m^n, f\right]_1$. Then for each $0 < \varepsilon < 1$, there exists a positive integer i_0 such that

$$\frac{1}{\lambda_i} \sum_{k \in I_i} a_{sk} f\left(\left\|\Delta_m^n x_k - L\right\|\right) \leq \varepsilon < 1 \text{ for all } i \geq i_0.$$

This implies that

$$\frac{1}{\lambda_i} \sum_{k \in I_i} a_{sk} \left[f\left(\left\|\Delta_m^n x_k - L\right\|\right)\right]^{p_k} \leq \frac{1}{\lambda_i} \sum_{k \in I_i} a_{sk} f\left(\left\|\Delta_m^n x_k - L\right\|\right).$$

Therefore $x \in \left[V_\lambda^E, A, \Delta_m^n, f, p\right]_1$.

(c) Using the same technique as in Theorem 2 of Nanda [14], it is easy to prove (c). □

3. Statistical convergence

The notion of statistical convergence was introduced by Fast [5] and studied by various authors. Recently, Mursaleen [12] introduced a new concept of statistical convergence as follows:

A sequence $x = (x_k)$ is said to be $\lambda-$statistically convergent or $s_\lambda-$statistically convergent to L if for every $\varepsilon > 0$

$$\lim_i \frac{1}{\lambda_i} \left|\left\{k \in I_i : \ |x_k - L| \geq \varepsilon\right\}\right| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set. In this case we write $s_\lambda - \lim x = L$ or $x_k \rightarrow L\,(s_\lambda)$ and $s_\lambda = \{\ x = (x_k)\colon \exists L \in \mathbb{R}, s_\lambda - \lim x = L\}$.

Definition 3.1. A sequence $x = (x_k)$ is said to be $\lambda_E^{n,m}-$statistically convergent to L if for every $\varepsilon > 0$

$$\lim_i \frac{1}{\lambda_i} \left|\left\{k \in I_i : \|\ \Delta_m^n x_k - L\| \geq \varepsilon\right\}\right| = 0.$$

In this case we write $\left[s_\lambda^E, \Delta_m^n\right] - \lim x = L$ or $x_k \to L\left(\left[s_\lambda^E, \Delta_m^n\right]\right)$. In the case $m = 0$, $\left[s_\lambda^E, \Delta_m^n\right]$ reduces to $s_\lambda\left(\Delta_m^n\right)$ which was studied by Et et al. [4].

Theorem 3.1. *Let $\lambda = (\lambda_i)$ be the same as in Section 1, then*
(a) *If $x_k \to L\left[V_\lambda^E, \Delta_m^n\right]_1$, then $x_k \to L\left(\left[s_\lambda^E, \Delta_m^n\right]\right)$, where*

$$\left[V_\lambda^E, \Delta_m^n\right]_1 = \left\{ x = (x_k) \in w(E) : \lim_i \frac{1}{\lambda_i} \sum_{k \in I_i} \left\| \Delta_m^n x_k - L \right\| = 0, \text{ for some } L \right\}.$$

(b) *If $x \in l_\infty\left(\Delta_m^n, E\right)$ and $x_k \to L\left(\left[s_\lambda^E, \Delta_m^n\right]\right)$, then $x_k \to L\left[V_\lambda^E, \Delta_m^n\right]_1$, where*

$$l_\infty\left(\Delta_m^n, E\right) = \left\{ x = (x_k) \in w(E) : \sup_k \left\| \Delta_m^n x_k \right\| < \infty \right\}.$$

(c) $\left[s_\lambda^E, \Delta_m^n\right] \cap l_\infty\left(\Delta_m^n, E\right) = \left[V_\lambda^E, \Delta_m^n\right]_1 \cap l_\infty\left(\Delta_m^n, E\right)$.

Proof. (a) Let $\varepsilon > 0$ and $x_k \to L\left[V_\lambda^E, \Delta_m^n\right]_1$, then we have

$$\frac{1}{\lambda_i} \sum_{k \in I_i} \left\| \Delta_m^n x_k - L \right\| \geq \varepsilon \left| \left\{ k \in I_i : \left\| \Delta_m^n x_k - L \right\| \geq \varepsilon \right\} \right|.$$

So, $x_k \to L\left(\left[s_\lambda^E, \Delta_m^n\right]\right)$. In fact, the set $\left[V_\lambda^E, \Delta_m^n\right]_1$ is a proper subset of $\left[s_\lambda^E, \Delta_m^n\right]$. To show this, let $E = \mathbb{C}$, and we define $x = (x_k)$ by $\Delta_m^n x_k = k$, for $i - \left[\left|\sqrt{i}\right|\right] + 1 \leq k \leq i$ and $\Delta_m^n x_k = 0$, otherwise. Then $x \notin l_\infty\left(\Delta_m^n, E\right)$ and $x \notin \left[V_\lambda^E, \Delta_m^n\right]_1$ but $x_k \to L = 0\left(\left[s_\lambda^E, \Delta_m^n\right]\right)$.

(b) Suppose that $x_k \to L\left(\left[s_\lambda^E, \Delta_m^n\right]\right)$ and $x \in l_\infty\left(\Delta_m^n, E\right)$, say $\left\| \Delta_m^n x_k - L \right\| \leq T$ $(T \geq 0)$. Given $\varepsilon > 0$, we have

$$\begin{aligned} \frac{1}{\lambda_i} \sum_{k \in I_i} \left\| \Delta_m^n x_k - L \right\| &= \frac{1}{\lambda_i} \sum_{\substack{k \in I_i \\ \left\| \Delta_m^n x_k - L \right\| \geq \varepsilon}} \left\| \Delta_m^n x_k - L \right\| \\ &\quad + \frac{1}{\lambda_i} \sum_{\substack{k \in I_i \\ \left\| \Delta_m^n x_k - L \right\| < \varepsilon}} \left\| \Delta_m^n x_k - L \right\| \\ &\leq \frac{T}{\lambda_i} \left| \left\{ k \in I_i : \left\| \Delta_m^n x_k - L \right\| \geq \varepsilon \right\} \right| + \varepsilon. \end{aligned}$$

Hence $x_k \to L\left[V_\lambda^E, \Delta_m^n\right]_1$.

(c) This immediately follows from (a) and (b). □

Proposition 3.2. *If $\lim_i \frac{\lambda_i}{i} > 0$, then $\left[s^E, \Delta_m^n\right] \subset \left[s_\lambda^E, \Delta_m^n\right]$, where*

$$\left[s^E, \Delta_m^n\right] = \left\{ x = (x_k) : \lim_i \frac{1}{i} \left| \left\{ k \leq i : \left\| \Delta_m^n x_k - L \right\| \geq \varepsilon \right\} \right| = 0 \right\}.$$

Proof. For given $\varepsilon > 0$, we get

$$\{k \le i : \| \Delta_m^n x_k - L\| \ge \varepsilon\} \supset \{k \in I_i : \| \Delta_m^n x_k - L\| \ge \varepsilon\}.$$

Hence

$$\begin{aligned} \frac{1}{i}\left|\{k \le i : \| \Delta_m^n x_k - L\| \ge \varepsilon\}\right| &\ge \frac{1}{i}\left|\{k \in I_i : \| \Delta_m^n x_k - L\| \ge \varepsilon\}\right| \\ &= \frac{\lambda_i}{i}\frac{1}{\lambda_i}\left|\{k \in I_i : \| \Delta_m^n x_k - L\| \ge \varepsilon\}\right|. \end{aligned}$$

So, we obtain $x \in \left[s_\lambda^E, \Delta_m^n\right]$. □

Proposition 3.3. *Let f be a modulus function, $a_{sk} = 1$, for all $s, k \in \mathbb{N}$ and $0 < h = \inf_k p_k \le p_k \le \sup_k p_k = H < \infty$. Then*

$$\left[V_\lambda^E, A, \Delta_m^n, f\right]_1 \subset \left[s_\lambda^E, \Delta_m^n\right].$$

Proof. Let $x \in \left[V_\lambda^E, A, \Delta_m^n, f\right]_1$ and $\varepsilon > 0$ be given.

$$\begin{aligned} \frac{1}{\lambda_i}\sum_{k\in I_i} f(\|\Delta_m^n x_k - L\|) &= \frac{1}{\lambda_i}\sum_{\substack{k \in I_i \\ \|\Delta_m^n x_k - L\| \ge \varepsilon}} f\left(\|\Delta_m^n x_k - L\|\right) \\ &\quad + \frac{1}{\lambda_i}\sum_{\substack{k \in I_i \\ \| \Delta_m^n x_k - L\| < \varepsilon}} f\left(\|\Delta_m^n x_k - L\|\right) \\ &\ge \frac{1}{\lambda_i}\sum_{\substack{k \in I_i \\ \| \Delta_m^n x_k - L\| \ge \varepsilon}} f\left(\|\Delta_m^n x_k - L\|\right) \\ &\ge \frac{1}{\lambda_i}\sum_{\substack{k \in I_i \\ \| \Delta_m^n x_k - L\| \ge \varepsilon}} f(\varepsilon) \\ &\ge \frac{1}{\lambda_i}\left|\{k \in I_i : \|\Delta_m^n x_k - L\| \ge \varepsilon\}\right|.f(\varepsilon). \end{aligned}$$

So, we obtain $x \in \left[s_\lambda^E, \Delta_m^n\right]$. □

Proposition 3.4. *Let f be bounded and $a_{sk} = 1$, for all $s, k \in \mathbb{N}$ and $0 < h = \inf_k p_k \le p_k \le \sup_k p_k = H < \infty$. Then*

$$\left[V_\lambda^E, A, \Delta_m^n, f\right]_1 \supset \left[s_\lambda^E, \Delta_m^n\right].$$

Proof. Suppose that f is bounded. Let $\varepsilon > 0$ be given. Since f is bounded, there exists an integer T such that $f(x) < T$ for all $x \geq 0$. Then

$$\begin{aligned}\frac{1}{\lambda_i}\sum_{k\in I_i} f\left(\|\Delta_m^n x_k - L\|\right) &= \frac{1}{\lambda_i}\sum_{\substack{k\in I_i \\ \|\Delta_m^n x_k - L\|\geq\varepsilon}} f\left(\|\Delta_m^n x_k - L\|\right) \\ &\quad + \frac{1}{\lambda_i}\sum_{\substack{k\in I_i \\ \|\Delta_m^n x_k - L\|<\varepsilon}} f\left(\|\Delta_m^n x_k - L\|\right) \\ &\leq \frac{1}{\lambda_i}\sum_{k\in I_i} T + \frac{1}{\lambda_i}\sum_{k\in I_i} f(\varepsilon) \\ &= \frac{T}{\lambda_i}\left|\left\{k\in I_i : \|\Delta_m^n x_k - L\|\geq\varepsilon\right\}\right| + f(\varepsilon).\end{aligned}$$

Hence $x \in \left[V_\lambda^E, A, \Delta_m^n, f\right]_1$. □

Theorem 3.5. *Let* $a_{sk} = 1$, *for all* $s, k \in \mathbb{N}$ *and* $0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$. *Then*

$$\left[V_\lambda^E, A, \Delta_m^n, f\right]_1 = \left[s_\lambda^E, \Delta_m^n\right] \quad \Leftrightarrow f \text{ is bounded.}$$

Proof. Let f be bounded, by Proposition 3.3 and Proposition 3.4, we have

$$\left[V_\lambda^E, A, \Delta_m^n, f\right]_1 = \left[s_\lambda^E, \Delta_m^n\right].$$

Conversely, suppose that f is unbounded. Then there exists a sequence $z = (z_k)$ of positive numbers with $f(z_k) = k^2$ for $k = 1, 2, ...$ If we choose $\Delta_m^n x_j = z_k$ for $j = k^2$, $j = 1, 2, ...$ and $\Delta_m^n x_j = 0$, otherwise, then we have

$$\frac{1}{\lambda_i}\left|\left\{k\in I_i : \ |\Delta_m^n x_k - L| \geq \varepsilon\right\}\right| \leq \frac{\sqrt{\lambda_{i-1}}}{\lambda_i} \text{ for all } i\in\mathbb{N}$$

and so $x \in \left[s_\lambda^{\mathbb{C}}, \Delta_m^n\right]$ but $x \notin \left[V_\lambda^{\mathbb{C}}, A, \Delta_m^n, f\right]_1$ for $E = \mathbb{C}$. This contradicts $\left[V_\lambda^E, A, \Delta_m^n, f\right]_1 = \left[s_\lambda^E, \Delta_m^n\right]$. □

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