ACTA MATHEMATICA VIETNAMICA
Volume 35, Number 2, 2010, pp. 207–227

AN N-ORDER ITERATIVE SCHEME FOR A NONLINEAR KIRCHHOFF-CARRIER WAVE EQUATION ASSOCIATED WITH MIXED HOMOGENEOUS CONDITIONS

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Abstract. In this paper, a high-order iterative scheme is established in order to get a convergent sequence at a rate of order N $(N \geq 1)$ to a local unique weak solution of a nonlinear Kirchhoff – Carrier wave equation associated with mixed homogeneous conditions. This extends recent corresponding results where recurrent sequences converge at a rate of order 1 or 2.

1. Introduction

In this paper we consider a nonlinear wave equation with the Kirchhoff-Carrier operator

(1.1)
$$u_{tt} - \mu\left(t, ||u(t)||^2, ||u_x(t)||^2\right) \frac{\partial}{\partial x}\left(A(x)u_x\right) = f(x,t,u),\ 0 < x < 1,\ 0 < t < T,$$

(1.2)
$$A(0)u_x(0,t) - hu(0,t) = u(1,t) = 0,$$

(1.3)
$$u(x,0) = \widetilde{u}_0(x),\ u_t(x,0) = \widetilde{u}_1(x),$$

where A, μ, f, $\widetilde{u}_0$, $\widetilde{u}_1$ are given functions satisfying conditions specified later and $h \geq 0$ is a given constant. In Eq. (1.1), the nonlinear term $\mu(t, ||u(t)||^2, ||u_x(t)||^2)$ depends on the integrals

(1.4)
$$||u(t)||^2 = \int_0^1 |u(x,t)|^2 dx,\ ||u_x(t)||^2 = \int_0^1 |u_x(x,t)|^2 dx.$$

Eq. (1.1) has its origin in the nonlinear vibration of an elastic string (Kirchhoff [5]), for which the associated equation is

(1.5)
$$\rho h u_{tt} = \left(P_0 + \frac{Eh}{2L}\int_0^L |\frac{\partial u}{\partial y}(y,t)|^2 dy\right) u_{xx},$$

here u is the lateral deflection, ρ is the mass density, h is the cross section, L is the length, E is Young's modulus and P_0 is the initial axial tension.

Received November 18, 2008; in revised form May 25, 2009.
2000 *Mathematics Subject Classification.* 35L20, 35L70, 35Q72.
Key words and phrases. Nonlinear Kirchhoff - Carrier wave equation, Faedo - Galerkin method, the convergence of order N.

In [3], Carrier also established a model of the type

$$u_{tt} = \left(P_0 + P_1 \int_0^L u^2(y,t)dy \right) u_{xx}, \tag{1.6}$$

where P_0 and P_1 are constants.

In [8] Long and Diem have studied the linear recursive schemes associated with the nonlinear wave equation

$$u_{tt} - u_{xx} = f(x,t,u,u_x,u_t),\ 0 < x < 1,\ 0 < t < T, \tag{1.7}$$

associated with (1.3) and the mixed conditions (1.2) standing for

$$u_x(0,t) - h_0 u(0,t) = u_x(1,t) + h_1 u(1,t) = 0, \tag{1.8}$$

where $h_0 > 0$, $h_1 \geq 0$ are given constants. This result has been extended in [9] to the nonlinear wave equation with the Kirchhoff operator

$$u_{tt} - \mu(||u_x||^2)u_{xx} = f(x,t,u,u_x,u_t),\ 0 < x < 1,\ 0 < t < T, \tag{1.9}$$

associated with (1.3) and the Dirichlet homogeneous boundary condition.

The authors of [8], [9] proved that there exists a recurrent sequence which converges at a rate of order 1 to a weak solution of the problem. Afterwards, the quadratic convergence also has been studied in [11] - [14].

Based on the ideas about recurrence relations for a third order method for solving the nonlinear operator equation $F(u) = 0$ in [15], we extend the above results by the construction a high-order iterative scheme.

In this paper, we associate with equation (1.1) a recurrent sequence $\{u_m\}$ defined by

$$\begin{aligned} &\frac{\partial^2 u_m}{\partial t^2} - \mu(t, ||u_m||^2, ||u_{mx}||^2) \frac{\partial}{\partial x}\left(A(x) u_{mx}\right) \\ &= \sum_{i=0}^{N-1} \frac{1}{i!} \frac{\partial^i f}{\partial u^i}(x,t,u_{m-1}) \left(u_m - u_{m-1}\right)^i, \end{aligned} \tag{1.10}$$

$0 < x < 1$, $0 < t < T$, with u_m satisfying (1.2), (1.3). The first term u_0 is chosen as $u_0 \equiv \widetilde{u}_0$. If $\mu \in C^1(\mathbb{R}_+^3)$, $A \in C^1([0,1])$, $A(x) \geq a_0 > 0$ and $f \in C^N([0,1] \times \mathbb{R}_+ \times \mathbb{R})$, we prove that the sequence $\{u_m\}$ converges at a rate of order N to a local unique weak solution of the problem (1.1) – (1.3). This result is a relative generalization of [2], [4], [8]-[14].

2. Preliminary results, notations

First, we denote the usual function spaces used in this paper by the notations $L^p = L^p(0,1)$, $H^m = H^m(0,1)$. Let $\langle \cdot, \cdot \rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $|| \cdot ||$ stands for the norm in L^2 and we denote by $|| \cdot ||_X$ the norm in the Banach space X. We call X' the dual space of X. We denote by

$L^p(0,T;X)$, $1 \leq p \leq \infty$ for the Banach space of real functions $u : (0,T) \to X$ measurable, such that

$$||u||_{L^p(0,T;X)} = \left(\int_0^T ||u(t)||_X^p dt\right)^{1/p} < +\infty \text{ for } 1 \leq p < \infty,$$

and

$$||u||_{L^\infty(0,T;X)} = \underset{0<t<T}{ess\sup} ||u(t)||_X \text{ for } p = \infty.$$

Let $u(t)$, $u_t(t) = \dot{u}(t)$, $u_{tt}(t) = \ddot{u}(t)$, $u_x(t) = \bigtriangledown u(t)$, $u_{xx}(t) = \Delta u(t)$, denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively. With $f \in C^k([0,1] \times \mathbb{R}_+ \times \mathbb{R})$, $f = f(x,t,u)$, we put $D_1 f = \frac{\partial f}{\partial x}$, $D_2 f = \frac{\partial f}{\partial t}$, $D_3 f = \frac{\partial f}{\partial u}$ and $D^\alpha f = D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3} f$, $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}_+^3$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 = k$.

Similarly, with $\mu = \mu(t,y,z)$, we also put $D_1 \mu = \frac{\partial \mu}{\partial t}$, $D_2 \mu = \frac{\partial \mu}{\partial y}$, $D_3 \mu = \frac{\partial \mu}{\partial z}$.

Next, let $A \in C([0,1])$, with $A(x) \geq a_0 > 0$ for all $x \in [0,1]$. We put

$$a(u,v) = \int_0^1 A(x) u_x(x) v_x(x) dx + h u(0) v(0), \tag{2.1}$$

$$V = \{v \in H^1 : v(1) = 0\}. \tag{2.2}$$

Then V is a closed subspace of H^1 and on V three norms $||v||_{H^1}$, $||v_x||$ and $||v||_a = \sqrt{a(v,v)}$ are equivalent norms.

Then we have the following lemmas, the proofs of which are straightforward and are omitted.

Lemma 2.1. *The imbedding $H^1 \hookrightarrow C^0([0,1])$ is compact and*

$$||v||_{C^0([0,1])} \leq \sqrt{2} ||v||_{H^1} \text{ for all } v \in H^1. \tag{2.3}$$

Lemma 2.2. *Let $h \geq 0$. Then the imbedding $V \hookrightarrow C^0([0,1])$ is compact and*

$$\begin{cases} ||v||_{C^0([0,1])} \leq ||v_x|| \leq \frac{1}{\sqrt{a_0}} ||v||_a, \\ \frac{1}{\sqrt{2}} ||v||_{H^1} \leq ||v_x|| \leq ||v||_{H^1}, \\ \sqrt{a_0} ||v_x|| \leq ||v||_a \leq \sqrt{A_{\max} + h} ||v_x||, \end{cases} \tag{2.4}$$

for all $v \in V$, where $A_{\max} = ||A||_{C^0([0,1])}$.

Lemma 2.3. *Let $h \geq 0$. Then the symmetric bilinear form $a(\cdot,\cdot)$ defined by (2.2) is continuous on $V \times V$ and coercive on V.*

Lemma 2.4. *Let $h \geq 0$. Then there exists the Hilbert orthonormal base $\{\widetilde{w}_j\}$ of L^2 consisting of the eigenfunctions $\widetilde{w}_j$ corresponding to the eigenvalue λ_j such that*

$$\begin{cases} 0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_j \leq ..., \ \lim_{j \to +\infty} \lambda_j = +\infty, \\ a(\widetilde{w}_j, v) = \lambda_j \langle \widetilde{w}_j, v \rangle \text{ for all } v \in V, \ j = 1, 2, ... \end{cases} \tag{2.5}$$

Furthermore, the sequence $\{\widetilde{w}_j / \sqrt{\lambda_j}\}$ is also the Hilbert orthonormal base of V with respect to the scalar product $a(\cdot,\cdot)$.

On the other hand, we also have $\widetilde{w}_j$ satisfying the following boundary value problem

$$\begin{cases} -\frac{\partial}{\partial x}\left(A(x)\frac{\partial \widetilde{w}_j}{\partial x}\right) = \lambda_j \widetilde{w}_j, \ in\ \Omega, \\ \frac{\partial \widetilde{w}_j}{\partial x}(0) - \frac{h}{A(0)}\widetilde{w}_j(0) = \widetilde{w}_j(1) = 0, \ \widetilde{w}_j \in C^\infty(\overline{\Omega}). \end{cases} \tag{2.6}$$

The proof of Lemma 2.4 can be found in [16, p.87, Theorem 7.7], with $H = L^2$ and V, $a(\cdot,\cdot)$ defined by (2.1), (2.2).

Finally, let us note more that the weak solution u of the initial and boundary value problem (1.1) – (1.3) will be obtained in Section 3 (Theorem 3.4) in the following manner:

Find $u \in \widetilde{W} = \left\{v \in L^\infty(0,T;V\cap H^2) : v_t \in L^\infty(0,T;V),\ v_{tt} \in L^\infty(0,T;L^2)\right\}$ such that u verifies the following variational equation

$$\langle u_{tt}(t), v\rangle + \mu\left(t, ||u(t)||^2, ||u_x(t)||^2\right) a(u(t),v) = \langle f(\cdot,t,u), v\rangle \ \ \forall v \in V, \tag{2.7}$$

and the initial conditions

$$u(0) = \widetilde{u}_0,\ u_t(0) = \widetilde{u}_1. \tag{2.8}$$

3. The N-order iterative scheme

We make the following assumptions:

(H_1) $h \geq 0$;

(H_2) $\widetilde{u}_0 \in V \cap H^2$ and $\widetilde{u}_1 \in V$;

(H_3) $A \in C^1([0,1])$ and there exists a constant $a_0 > 0$ such that $A(x) \geq a_0$ for all $x \in [0,1]$;

(H_4) $\mu \in C^1(\mathbb{R}^3_+)$ and there exist constants $p > 1$, $\mu_* > 0$, $\mu_i > 0$, $i \in \{0,1,2,3\}$, such that

(i) $\mu_* \leq \mu(t,y,z) \leq \mu_0\left(1 + y^p + z^p\right)$, for all $(t,y,z) \in \mathbb{R}^3_+$,

(ii) $|D_1\mu(t,y,z)| \leq \mu_1\left(1 + y^p + z^p\right)$, for all $(t,y,z) \in \mathbb{R}^3_+$,

(iii) $|D_2\mu(t,y,z)| \leq \mu_2\left(1 + y^{p-1} + z^p\right)$, for all $(t,y,z) \in \mathbb{R}^3_+$,

(iv) $|D_3\mu(t,y,z)| \leq \mu_3\left(1 + y^p + z^{p-1}\right)$, for all $(t,y,z) \in \mathbb{R}^3_+$;

(H_5) $f \in C^N([0,1] \times \mathbb{R}_+ \times \mathbb{R})$.

With f satisfying the assumption (H_5), for each $M > 0$ and $T > 0$ we put

$$\begin{cases} K_0 = K_0(M,T,f) = \sup\left\{|f(x,t,u)| : (x,t,u) \in A_*\right\}, \\ K_i = K_i(M,T,f) = \sum_{|\alpha|=i} K_0(M,T,D^\alpha f), \\ \widehat{K}_i = \max\limits_{0\leq j\leq i} K_j, \end{cases} \tag{3.1}$$

$i = 1,2,...,N$, where

$$A_* = A_*(M,T) = \{(x,t,u) \in \mathbb{R}^3 : 0 \leq x \leq 1,\ 0 \leq t \leq T,\ |u| \leq M\}.$$

For each $M > 0$ and $T > 0$ we get

(3.2)

$$\begin{cases} W(M,T) = \{v \in L^\infty(0,T;V\cap H^2) : v_t \in L^\infty(0,T;V) \text{ and } v_{tt} \in L^2(Q_T), \\ \qquad\qquad \text{with } ||v||_{L^\infty(0,T;V\cap H^2)},\ ||v_t||_{L^\infty(0,T;V)},\ ||v_{tt}||_{L^2(Q_T)} \le M\}, \\ W_1(M,T) = \{v \in W(M,T) : v_{tt} \in L^\infty(0,T;L^2)\}. \end{cases}$$

We shall choose as first initial term $u_0 \equiv \widetilde{u}_0$, suppose that

$$u_{m-1} \in W_1(M,T), \tag{3.3}$$

and associate with problem (1.4), (1.6), (1.7) the following variational problem:

Find $u_m \in W_1(M,T)$ $(m \ge 1)$ so that

$$\begin{cases} \langle \ddot{u}_m(t), v\rangle + \mu_m(t) a(u_m(t), v) = \langle F_m(t), v\rangle \ \forall v \in V, \\ u_m(0) = \widetilde{u}_0,\ \dot{u}_m(0) = \widetilde{u}_1, \end{cases} \tag{3.4}$$

where

$$\mu_m(t) = \mu\left(t, ||u_m(t)||^2, ||u_{mx}(t)||^2\right), \tag{3.5}$$

$$F_m(x,t) = \textstyle\sum_{i=0}^{N-1} \frac{1}{i!} D_3^i f(x,t,u_{m-1}) \left(u_m - u_{m-1}\right)^i. \tag{3.6}$$

Then, we have the following theorem.

Theorem 3.1. *Let $(H_1) - (H_5)$ hold. Then there exist a constant $M > 0$ depending on A, $\widetilde{u}_0$, $\widetilde{u}_1$, μ and a constant $T > 0$ depending on A, $\widetilde{u}_0$, $\widetilde{u}_1$, μ, f such that, for $u_0 \equiv \widetilde{u}_0$, there exists a recurrent sequence $\{u_m\} \subset W_1(M,T)$ defined by* (3.4)– (3.6).

Proof. The proof consists of several steps.

Step 1: *The Faedo-Galerkin approximation* (introduced by Lions [7]). Consider the basis for V as in Lemma 2.4 ($w_j = \widetilde{w}_j/\sqrt{\lambda_j}$). Put

$$u_m^{(k)}(t) = \textstyle\sum_{j=1}^k c_{mj}^{(k)}(t) w_j, \tag{3.7}$$

where the coefficients $c_{mj}^{(k)}$ satisfy the system of nonlinear differential equations

$$\begin{cases} \langle \ddot{u}_m^{(k)}(t), w_j\rangle + \mu_m^{(k)}(t) a(u_m^{(k)}(t), w_j) = \langle F_m^{(k)}(t), w_j\rangle,\ 1 \le j \le k, \\ u_m^{(k)}(0) = \widetilde{u}_{0k},\ \dot{u}_m^{(k)}(0) = \widetilde{u}_{1k}, \end{cases} \tag{3.8}$$

where

$$\begin{cases} \widetilde{u}_{0k} = \sum_{j=1}^k \alpha_j^{(k)} w_j \to \widetilde{u}_0 \text{ strongly in } H^2, \\ \widetilde{u}_{1k} = \sum_{j=1}^k \beta_j^{(k)} w_j \to \widetilde{u}_1 \text{ strongly in } H^1, \end{cases} \tag{3.9}$$

and

$$\begin{cases} \mu_m^{(k)}(t) = \mu\left(t, ||u_m^{(k)}(t)||^2, ||\nabla u_m^{(k)}(t)||^2\right), \\ F_m^{(k)}(x,t) = \sum_{i=0}^{N-1} \frac{1}{i!} D_3^i f(x,t,u_{m-1}) \left(u_m^{(k)} - u_{m-1}\right)^i. \end{cases} \tag{3.10}$$

Let us suppose that u_{m-1} satisfies (3.3). Then it is clear that system (3.8) has a solution $u_m^{(k)}(t)$ on an interval $0 \leq t \leq T_m^{(k)} \leq T$. The following estimates allow one to take constant $T_m^{(k)} = T$ for all m and k.

Step 2: *A priori estimates.* Put

$$(3.11) \qquad \begin{cases} f_1(t) = f(1,t,0), \\ s_m^{(k)}(t) = ||\dot{u}_m^{(k)}(t)||^2 + ||\dot{u}_m^{(k)}(t)||_a^2 \\ \qquad\qquad +\mu_* \left(||u_m^{(k)}(t)||_a^2 + ||\frac{\partial}{\partial x}\left(A\frac{\partial u_m^{(k)}}{\partial x}(t)\right)||^2 \right) + \int_0^t ||\ddot{u}_m^{(k)}(s)||^2 ds, \\ S_m^{(k)}(t) = X_m^{(k)}(t) + Y_m^{(k)}(t) + \int_0^t ||\ddot{u}_m^{(k)}(s)||^2 ds, \end{cases}$$

where

$$(3.12) \qquad \begin{cases} X_m^{(k)}(t) = ||\dot{u}_m^{(k)}(t)||^2 + \mu_m^{(k)}(t)||u_m^{(k)}(t)||_a^2, \\ Y_m^{(k)}(t) = ||\dot{u}_m^{(k)}(t)||_a^2 + \mu_m^{(k)}(t)||\frac{\partial}{\partial x}\left(A\frac{\partial u_m^{(k)}}{\partial x}(t)\right)||^2. \end{cases}$$

Then, it follows from (3.8)-(3.12) that

$$(3.13) \qquad \begin{aligned} S_m^{(k)}(t) &= S_m^{(k)}(0) + \int_0^t \dot{\mu}_m^{(k)}(s) \left[||u_m^{(k)}(s)||_a^2 + ||\frac{\partial}{\partial x}\left(A\frac{\partial u_m^{(k)}}{\partial x}(s)\right)||^2 \right] ds \\ &\quad +2\int_0^t \langle F_m^{(k)}(s), \dot{u}_m^{(k)}(s)\rangle ds + 2\int_0^t a(F_m^{(k)}(s), \dot{u}_m^{(k)}(s)) ds \\ &\quad -2A(1)\int_0^t f_1(s)\nabla\dot{u}_m^{(k)}(1,s)ds + \int_0^t ||\ddot{u}_m^{(k)}(s)||^2 ds \\ &= S_m^{(k)}(0) + \sum_{j=1}^5 I_j. \end{aligned}$$

We shall estimate respectively the following terms on the right-hand side of (3.13).

First term I_1 : By $(3.10)_1$, we have

$$(3.14) \qquad \begin{aligned} \dot{\mu}_m^{(k)}(t) &= D_1\mu\left(t, ||u_m^{(k)}(t)||^2, ||\nabla u_m^{(k)}(t)||^2\right) \\ &\quad +2D_2\mu\left(t, ||u_m^{(k)}(t)||^2, ||\nabla u_m^{(k)}(t)||^2\right) \langle u_m^{(k)}(t), \dot{u}_m^{(k)}(t)\rangle \\ &\quad +2D_3\mu\left(t, ||u_m^{(k)}(t)||^2, ||\nabla u_m^{(k)}(t)||^2\right) \langle \nabla u_m^{(k)}(t), \nabla\dot{u}_m^{(k)}(t)\rangle. \end{aligned}$$

By using the assumption (H_4, (ii), (iii), (iv)), and the following inequalities

$$(3.15) \qquad \begin{aligned} ||u_m^{(k)}(t)|| &\leq ||u_m^{(k)}(t)||_{C^0([0,1])} \leq ||\nabla u_m^{(k)}(t)|| \\ &\leq \frac{1}{\sqrt{a_0}}||u_m^{(k)}(t)||_a \leq \frac{1}{\sqrt{a_0\mu_*}}\sqrt{s_m^{(k)}(t)}, \end{aligned}$$

$$(3.16) \qquad ||\nabla u_m^{(k)}(t)|| \leq \frac{1}{\sqrt{a_0}}||u_m^{(k)}(t)||_a \leq \frac{1}{\sqrt{a_0\mu_*}}\sqrt{s_m^{(k)}(t)},$$

$$(3.17) \qquad ||\dot{u}_m^{(k)}(t)|| \leq \sqrt{s_m^{(k)}(t)},$$

$$||\nabla \dot{u}_m^{(k)}(t)|| \leq \frac{1}{\sqrt{a_0}}||\dot{u}_m^{(k)}(t)||_a \leq \frac{1}{\sqrt{a_0}}\sqrt{s_m^{(k)}(t)}, \tag{3.18}$$

we deduce from (3.14), that

$$\begin{aligned}
|\dot{\mu}_m^{(k)}(t)| &\leq \mu_1 \left(1 + ||u_m^{(k)}(t)||^{2p} + ||\bigtriangledown u_m^{(k)}(t)||^{2p}\right) \\
&\quad + 2\mu_2 \left(1 + ||u_m^{(k)}(t)||^{2p-2} + ||\bigtriangledown u_m^{(k)}(t)||^{2p}\right) ||u_m^{(k)}(t)|| ||\dot{u}_m^{(k)}(t)|| \\
&\quad + 2\mu_3 \left(1 + ||u_m^{(k)}(t)||^{2p} + ||\bigtriangledown u_m^{(k)}(t)||^{2p-2}\right) ||\bigtriangledown u_m^{(k)}(t)|| ||\bigtriangledown \dot{u}_m^{(k)}(t)|| \\
&\leq \mu_1 \left[1 + \frac{2}{a_0^p \mu_*^p}\left(s_m^{(k)}(t)\right)^p\right] \\
&\quad + 2\mu_2 \left[1 + \frac{1}{a_0^{p-1}\mu_*^{p-1}}\left(s_m^{(k)}(t)\right)^{p-1} + \frac{1}{a_0^p \mu_*^p}\left(s_m^{(k)}(t)\right)^p\right] \frac{1}{\sqrt{a_0 \mu_*}} s_m^{(k)}(t) \\
&\quad + 2\mu_3 \left[1 + \frac{1}{a_0^p \mu_*^p}\left(s_m^{(k)}(t)\right)^p + \frac{1}{a_0^{p-1}\mu_*^{p-1}}\left(s_m^{(k)}(t)\right)^{p-1}\right] \frac{1}{\sqrt{\mu_*} a_0} s_m^{(k)}(t) \\
&= \mu_1 + 2\mu_2 + 2\mu_3 + \left[\frac{2\mu_1}{a_0^p \mu_*^p} + 2\left(\mu_2 + \frac{\mu_3}{\sqrt{a_0}}\right)\left(\frac{1}{a_0 \mu_*}\right)^{p-\frac{1}{2}}\right]\left(s_m^{(k)}(t)\right)^p \\
&\quad + \left[2\left(\mu_2 + \frac{\mu_3}{\sqrt{a_0}}\right)\left(\frac{1}{a_0 \mu_*}\right)^{p+\frac{1}{2}}\right]\left(s_m^{(k)}(t)\right)^{p+1} \\
&\leq \widetilde{\mu}_1 \left(1 + \left(s_m^{(k)}(t)\right)^p + \left(s_m^{(k)}(t)\right)^{p+1}\right), \qquad (3.19)
\end{aligned}$$

where

$$\widetilde{\mu}_1 = \mu_1 + 2\mu_2 + 2\mu_3 + \frac{2\mu_1}{a_0^p \mu_*^p} + 2\left(\mu_2 + \frac{\mu_3}{\sqrt{a_0}}\right)\left(1 + \frac{1}{a_0 \mu_*}\right)\left(\frac{1}{a_0 \mu_*}\right)^{p-\frac{1}{2}}. \tag{3.20}$$

Using the inequality

$$s^q \leq 1 + s^{N_0}, \ \forall s \geq 0, \ \forall q \in (0, N_0], \ N_0 = \max\{N-1, \ 2p+1\}, \tag{3.21}$$

we get from (3.11), (3.12), (3.19), that

$$\begin{aligned}
I_1 &= \int_0^t \dot{\mu}_m^{(k)}(s)\left[||u_m^{(k)}(s)||_a^2 + ||\frac{\partial}{\partial x}\left(A\frac{\partial u_m^{(k)}}{\partial x}(s)\right)||^2\right] ds \\
&\leq \widetilde{\mu}_1 \int_0^t \left(1 + \left(s_m^{(k)}(s)\right)^p + \left(s_m^{(k)}(s)\right)^{p+1}\right)\frac{1}{\mu_*} s_m^{(k)}(s) ds \\
&\leq \frac{\widetilde{\mu}_1}{\mu_*}\int_0^t \left(s_m^{(k)}(s) + \left(s_m^{(k)}(s)\right)^{p+1} + \left(s_m^{(k)}(s)\right)^{p+2}\right) ds \\
&\leq \frac{3\widetilde{\mu}_1}{\mu_*}\int_0^t \left[1 + \left(s_m^{(k)}(s)\right)^{N_0}\right] ds
\end{aligned}$$

$$\leq \frac{3\widetilde{\mu}_1}{\mu_*}\left[T+\int_0^t \left(s_m^{(k)}(s)\right)^{N_0} ds\right]. \tag{3.22}$$

We shall now require the following lemma.

Lemma 3.2. *We have*

$$||F_m^{(k)}(t)|| \leq \widehat{K}_{N-1}\textstyle\sum_{i=0}^{N-1}\widetilde{a}_i\left(\sqrt{s_m^{(k)}(t)}\right)^i, \tag{3.23}$$

$$||\nabla F_m^{(k)}(t)|| \leq \widetilde{K}_{N-1}\textstyle\sum_{i=0}^{N-1}\widetilde{a}_i\left(\sqrt{s_m^{(k)}(t)}\right)^i, \tag{3.24}$$

where $\widetilde{K}_N = (1+M)\widehat{K}_N + (N-1)\widehat{K}_{N-1}$, *with* $\widetilde{a}_i$, $i = 0,1,...,N-1$ *defined as follows*

$$\widetilde{a}_0 = 1+\tfrac{1}{2}\textstyle\sum_{i=1}^{N-1}\frac{(2M)^i}{i!},\ \widetilde{a}_i = \frac{1}{2i!}\left(\frac{2}{\sqrt{a_0\mu_*}}\right)^i,\ i=1,...,N-1. \tag{3.25}$$

Proof. (i) By (2.4), (3.3), $(3.10)_2$, (3.15), and (3.16), we have

$$\begin{aligned}
|F_m^{(k)}(x,t)| &\leq \widehat{K}_{N-1}+\widehat{K}_{N-1}\sum_{i=1}^{N-1}\frac{1}{i!}\left(||u_m^{(k)}(t)||_{C^0([0,1])}+||u_{m-1}(t)||_{C^0([0,1])}\right)^i\\
&\leq \widehat{K}_{N-1}+\widehat{K}_{N-1}\sum_{i=1}^{N-1}\frac{1}{i!}\left(||\nabla u_m^{(k)}(t)||+||\nabla u_{m-1}(t)||\right)^i\\
&\leq \widehat{K}_{N-1}+\widehat{K}_{N-1}\sum_{i=1}^{N-1}\frac{1}{i!}\left(\frac{1}{\sqrt{a_0\mu_*}}\sqrt{s_m^{(k)}(t)}+M\right)^i\\
&\leq \widehat{K}_{N-1}\left[1+\sum_{i=1}^{N-1}\frac{1}{i!}2^{i-1}\left(\left(\frac{1}{\sqrt{a_0\mu_*}}\right)^i\left(\sqrt{s_m^{(k)}(t)}\right)^i+M^i\right)\right]\\
&= \widehat{K}_{N-1}\left[1+\frac{1}{2}\sum_{i=1}^{N-1}\frac{(2M)^i}{i!}+\sum_{i=1}^{N-1}\frac{1}{2i!}\left(\frac{2}{\sqrt{a_0\mu_*}}\right)^i\left(\sqrt{s_m^{(k)}(t)}\right)^i\right]\\
&= \widehat{K}_{N-1}\sum_{i=0}^{N-1}\widetilde{a}_i\left(\sqrt{s_m^{(k)}(t)}\right)^i, \qquad (3.26)
\end{aligned}$$

with $\widetilde{a}_i$, $i=0,1,...,N-1$ defined as (3.25). Hence, (3.23) is proved.

(ii) We use the following notations: $f[u] = f(x,t,u)$, $D_jf[u] = D_jf(x,t,u)$, $j = 1,2,3$.

By $(3.10)_2$, we have

$$\begin{aligned}
\nabla F_m^{(k)}(x,t) &= D_1f[u_{m-1}] + D_3f[u_{m-1}]\nabla u_{m-1}\\
&+\sum_{i=1}^{N-1}\frac{1}{i!}\left(D_1D_3^if[u_{m-1}]+D_3^{i+1}f[u_{m-1}]\nabla u_{m-1}\right)\left(u_m^{(k)}-u_{m-1}\right)^i
\end{aligned}$$

$$+\sum_{i=1}^{N-1}\frac{i}{i!}D_3^i f[u_{m-1}]\left(u_m^{(k)}-u_{m-1}\right)^{i-1}\left(\nabla u_m^{(k)}-\nabla u_{m-1}\right). \tag{3.27}$$

Using the inequalities (2.4), (3.15), (2.6), it follows from (3.1), (3.3), (3.27), that

$$\begin{aligned}
&|\nabla F_m^{(k)}(x,t)| \\
&\leq K_1(1+|\nabla u_{m-1}|) \\
&\quad+\sum_{i=1}^{N-1}\frac{1}{i!}K_{i+1}\left(1+|\nabla u_{m-1}|\right)\left(||u_m^{(k)}(t)||_{C^0([0,1])}+||u_{m-1}(t)||_{C^0([0,1])}\right)^i \\
&\quad+\sum_{i=1}^{N-1}\frac{i}{i!}K_i\left(||u_m^{(k)}(t)||_{C^0([0,1])}+||u_{m-1}(t)||_{C^0([0,1])}\right)^{i-1}\left(|\nabla u_m^{(k)}|+|\nabla u_{m-1}|\right) \\
&\leq K_1(1+|\nabla u_{m-1}|)+\sum_{i=1}^{N-1}\frac{1}{i!}K_{i+1}\left(1+|\nabla u_{m-1}|\right)\left(\frac{1}{\sqrt{a_0\mu_*}}\sqrt{s_m^{(k)}(t)}+M\right)^i \\
&\quad+\sum_{i=1}^{N-1}\frac{i}{i!}K_i\left(\frac{1}{\sqrt{a_0\mu_*}}\sqrt{s_m^{(k)}(t)}+M\right)^{i-1}\left(|\nabla u_m^{(k)}|+|\nabla u_{m-1}|\right) \\
&\leq K_1(1+|\nabla u_{m-1}|)+\sum_{i=1}^{N-1}\frac{1}{i!}K_{i+1}\left(1+|\nabla u_{m-1}|\right)\left(\frac{1}{\sqrt{a_0\mu_*}}\sqrt{s_m^{(k)}(t)}+M\right)^i \\
&\quad+\sum_{i=1}^{N-1}\frac{i}{i!}K_i\left(\frac{1}{\sqrt{a_0\mu_*}}\sqrt{s_m^{(k)}(t)}+M\right)^{i-1}\left(|\nabla u_m^{(k)}|+|\nabla u_{m-1}|\right).
\end{aligned} \tag{3.28}$$

It follows from (2.4), (3.1), (3.3), (3.15), (3.16) and (3.28), that

$$\begin{aligned}
||\nabla F_m^{(k)}(t)|| &\leq K_1(1+M)+\sum_{i=1}^{N-1}\frac{1}{i!}K_{i+1}(1+M)\left(\frac{1}{\sqrt{a_0\mu_*}}\sqrt{s_m^{(k)}(t)}+M\right)^i \\
&\quad+\sum_{i=1}^{N-1}\frac{i}{i!}K_i\left(\frac{1}{\sqrt{a_0\mu_*}}\sqrt{s_m^{(k)}(t)}+M\right)^{i-1}\left(\frac{1}{\sqrt{a_0\mu_*}}\sqrt{s_m^{(k)}(t)}+M\right) \\
&\leq K_1(1+M)+\sum_{i=1}^{N-1}\frac{1}{i!}K_{i+1}(1+M)\left(\frac{1}{\sqrt{a_0\mu_*}}\sqrt{s_m^{(k)}(t)}+M\right)^i \\
&\quad+\sum_{i=1}^{N-1}\frac{i}{i!}K_i\left(\frac{1}{\sqrt{a_0\mu_*}}\sqrt{s_m^{(k)}(t)}+M\right)^{i} \\
&\leq (1+M)\widehat{K}_N\left[1+\sum_{i=1}^{N-1}\frac{1}{i!}\left(\frac{1}{\sqrt{a_0\mu_*}}\sqrt{s_m^{(k)}(t)}+M\right)^i\right]
\end{aligned}$$

$$
\begin{aligned}
&+ (N-1)\widehat{K}_{N-1}\sum_{i=1}^{N-1}\frac{1}{i!}\left(\frac{1}{\sqrt{a_0\mu_*}}\sqrt{s_m^{(k)}(t)}+M\right)^i \\
&\le \left[(1+M)\widehat{K}_N+(N-1)\widehat{K}_{N-1}\right]\left[1+\sum_{i=1}^{N-1}\frac{1}{i!}\left(\frac{1}{\sqrt{a_0\mu_*}}\sqrt{s_m^{(k)}(t)}+M\right)^i\right] \\
&= \widetilde{K}_N\left[1+\sum_{i=1}^{N-1}\frac{1}{i!}\left(\frac{1}{\sqrt{a_0\mu_*}}\sqrt{s_m^{(k)}(t)}+M\right)^i\right] \\
&\le \widetilde{K}_N\left[1+\sum_{i=1}^{N-1}\frac{1}{i!}2^{i-1}\left(\frac{1}{\left(\sqrt{a_0\mu_*}\right)^i}\left(\sqrt{s_m^{(k)}(t)}\right)^i+M^i\right)\right] \\
&= \widetilde{K}_N\left[1+\frac{1}{2}\sum_{i=1}^{N-1}\frac{(2M)^i}{i!}+\sum_{i=1}^{N-1}\frac{1}{2i!}\left(\frac{2}{\sqrt{a_0\mu_*}}\right)^i\left(\sqrt{s_m^{(k)}(t)}\right)^i\right] \\
&= \widetilde{K}_N\sum_{i=0}^{N-1}\widetilde{a}_i\left(\sqrt{s_m^{(k)}(t)}\right)^i. \qquad (3.29)
\end{aligned}
$$

Hence, (3.24) is proved. The proof of Lemma 3.2 is complete. □

We now return to the estimates for I_2, I_3.

Second term I_2 : We again use inequality (3.21) and from (3.17), (3.23), we have

$$
\begin{aligned}
I_2 &= 2\int_0^t \langle F_m^{(k)}(s), \dot{u}_m^{(k)}(s)\rangle ds \\
&\le 2\int_0^t ||F_m^{(k)}(s)||||\dot{u}_m^{(k)}(s)||ds \\
&= 2\widehat{K}_{N-1}\sum_{i=0}^{N-1}\widetilde{a}_i\int_0^t\left(\sqrt{s_m^{(k)}(s)}\right)^{i+1}ds \\
&\le 2\widehat{K}_{N-1}\sum_{i=0}^{N-1}\widetilde{a}_i\int_0^t\left[1+\left(s_m^{(k)}(s)\right)^{N_0}\right]ds \\
&\le 2\widehat{K}_{N-1}\sum_{i=0}^{N-1}\widetilde{a}_i\left[T+\int_0^t\left(s_m^{(k)}(s)\right)^{N_0}ds\right]. \qquad (3.30)
\end{aligned}
$$

Third term I_3 : We have

$$
\begin{aligned}
I_3 &= 2\int_0^t a(F_m^{(k)}(s), \dot{u}_m^{(k)}(s))ds \le 2\int_0^t ||F_m^{(k)}(s)||_a||\dot{u}_m^{(k)}(s)||_a ds \\
&\le 2\int_0^t ||F_m^{(k)}(s)||_a\sqrt{s_m^{(k)}(s)}ds. \qquad (3.31)
\end{aligned}
$$

On the other hand, by (2.3), (3.23) and (3.24), we obtain

$$\begin{aligned}
||F_m^{(k)}(t)||_a &= \sqrt{h|F_m^{(k)}(0,t)|^2 + \int_0^1 A(x)|\nabla F_m^{(k)}(x,t)|^2 dx} \\
&\leq \sqrt{2h||F_m^{(k)}(t)||_{H^1}^2 + A_{\max}||\nabla F_m^{(k)}(t)||^2} \\
&\leq \sqrt{(2h + A_{\max})\left(||F_m^{(k)}(t)||^2 + ||\nabla F_m^{(k)}(t)||^2\right)} \\
&\leq \sqrt{2h + A_{\max}}\left(||F_m^{(k)}(t)|| + ||\nabla F_m^{(k)}(t)||\right) \\
&\leq \sqrt{2h + A_{\max}}\left(\widehat{K}_{N-1} + \widetilde{K}_{N-1}\right)\sum_{i=0}^{N-1}\widetilde{a}_i\left(\sqrt{s_m^{(k)}(t)}\right)^i. \qquad (3.32)
\end{aligned}$$

Hence, it follows from (3.21), (3.31), (3.32), that

$$\begin{aligned}
I_3 &\leq 2\int_0^t ||F_m^{(k)}(s)||_a \sqrt{s_m^{(k)}(s)}ds \\
&\leq 2\sqrt{2h + A_{\max}}\left(\widehat{K}_{N-1} + \widetilde{K}_{N-1}\right)\sum_{i=0}^{N-1}\widetilde{a}_i\int_0^t\left(\sqrt{s_m^{(k)}(s)}\right)^{i+1} ds \\
&\leq 2\sqrt{2h + A_{\max}}\left(\widehat{K}_{N-1} + \widetilde{K}_{N-1}\right)\sum_{i=0}^{N-1}\widetilde{a}_i\int_0^t\left[1 + \left(s_m^{(k)}(s)\right)^{N_0}\right] ds \\
&\leq 2\sqrt{2h + A_{\max}}\left(\widehat{K}_{N-1} + \widetilde{K}_{N-1}\right)\sum_{i=0}^{N-1}\widetilde{a}_i\left[T + \int_0^t\left(s_m^{(k)}(s)\right)^{N_0} ds\right]. \qquad (3.33)
\end{aligned}$$

Fourth term I_4 : Integrating by parts, we have

$$\begin{aligned}
I_4 &= -2A(1)\textstyle\int_0^t f_1(s)\nabla \dot{u}_m^{(k)}(1,s)ds \\
&= -2A(1)f_1(t)\nabla u_m^{(k)}(1,t) + 2A(1)f_1(0)\nabla\widetilde{u}_{0k}(1) \\
&\quad + 2A(1)\textstyle\int_0^t f_1'(s)\nabla u_m^{(k)}(1,s)ds \qquad (3.34) \\
&= -2A(1)\left(\textstyle\int_0^t f_1'(s)ds\right)\nabla u_m^{(k)}(1,t) - 2A(1)f_1(0)\nabla u_m^{(k)}(1,t) \\
&\quad + 2A(1)f_1(0)\nabla\widetilde{u}_{0k}(1) + 2A(1)\textstyle\int_0^t f_1'(s)\nabla u_m^{(k)}(1,s)ds.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\nabla u_m^{(k)}(1,t) &= \nabla u_m^{(k)}(0,t) + \int_0^1 \Delta u_m^{(k)}(x,t)dx \\
&= \frac{h}{A(0)}u_m^{(k)}(0,t) + \int_0^1 \Delta u_m^{(k)}(x,t)dx, \qquad (3.35)
\end{aligned}$$

and

$$a_0||\Delta u_m^{(k)}(t)|| \leq ||A\Delta u_m^{(k)}(t)||$$

$$
\begin{aligned}
&= ||\frac{\partial}{\partial x}\left(A\frac{\partial u_m^{(k)}}{\partial x}(t)\right) - \nabla A \nabla u_m^{(k)}(t)|| \\
&\leq ||\frac{\partial}{\partial x}\left(A\frac{\partial u_m^{(k)}}{\partial x}(t)\right)|| + ||\nabla A||_{L^\infty(\Omega)}||\nabla u_m^{(k)}(t)|| \\
&\leq \frac{1}{\sqrt{\mu_*}}\sqrt{s_m^{(k)}(t)} + ||\nabla A||_{L^\infty(\Omega)}\frac{1}{\sqrt{a_0\mu_*}}\sqrt{s_m^{(k)}(t)} \\
&= \frac{1}{\sqrt{\mu_*}}\left(1 + \frac{1}{\sqrt{a_0}}||\nabla A||_{L^\infty(\Omega)}\right)\sqrt{s_m^{(k)}(t)}. \qquad (3.36)
\end{aligned}
$$

Hence, we obtain from (3.16), (3.35), (3.36), that

$$
\begin{aligned}
|\nabla u_m^{(k)}(1,t)|^2 &\leq \frac{2h^2}{A^2(0)}|u_m^{(k)}(0,t)|^2 + 2||\Delta u_m^{(k)}(t)||^2 \\
&\leq \frac{2h^2}{A^2(0)}||\nabla u_m^{(k)}(t)||^2 + 2||\Delta u_m^{(k)}(t)||^2 \\
&= \frac{2}{a_0\mu_*}\left[\frac{h^2}{A^2(0)} + \frac{1}{a_0}\left(1 + \frac{1}{\sqrt{a_0}}||\nabla A||_{L^\infty(\Omega)}\right)^2\right]s_m^{(k)}(t) \\
&= \widetilde{\mu}_4 s_m^{(k)}(t), \qquad (3.37)
\end{aligned}
$$

where

$$
\widetilde{\mu}_4 = \tfrac{2}{a_0\mu_*}\left[\tfrac{h^2}{A^2(0)} + \tfrac{1}{a_0}\left(1 + \tfrac{1}{\sqrt{a_0}}\left\|\nabla A\right\|_{L^\infty(\Omega)}\right)^2\right]. \qquad (3.38)
$$

It follows from (3.34), (3.37), that

$$
\begin{aligned}
|I_4| &\leq 2A(1)\left(\int_0^t \left|f_1'(s)\right| ds\right)\sqrt{\widetilde{\mu}_4}\sqrt{s_m^{(k)}(t)} + 2A(1)\left|f_1(0)\right|\sqrt{\widetilde{\mu}_4}\sqrt{s_m^{(k)}(t)} \\
&\quad + 2A(1)\left|f_1(0)\nabla\widetilde{u}_{0k}(1)\right| + 2A(1)\sqrt{\widetilde{\mu}_4}\int_0^t \left|f_1'(s)\right|\sqrt{s_m^{(k)}(s)}ds \\
&\leq 2\beta s_m^{(k)}(t) + \frac{1}{\beta}A^2(1)\widetilde{\mu}_4\left(T^2||f_1'||_{L^\infty}^2 + f_1^2(0)\right) \\
&\quad + 2A(1)\left|f_1(0)\nabla\widetilde{u}_{0k}(1)\right| + 2A(1)\sqrt{\widetilde{\mu}_4}||f_1'||_{L^\infty}^2\left[T + \int_0^t \left(s_m^{(k)}(s)\right)^{N_0} ds\right], \qquad (3.39)
\end{aligned}
$$

for all $\beta > 0$.

Fifth term I_5 : Equation $(3.8)_1$ can be rewritten as follows

$$
\langle \ddot{u}_m^{(k)}(t), w_j\rangle - \mu_m^{(k)}(t)\langle\tfrac{\partial}{\partial x}\left(A\tfrac{\partial u_m^{(k)}}{\partial x}(t)\right), w_j\rangle = \langle F_m^{(k)}(t), w_j\rangle,\ 1 \leq j \leq k. \qquad (3.40)
$$

Hence, it follows after replacing w_j with $\ddot{u}_m^{(k)}(t)$ and integrating that

$$\begin{aligned} I_5 &= \int_0^t ||\ddot{u}_m^{(k)}(s)||^2 ds \\ &\leq 2\int_0^t ||F_m^{(k)}(s)||^2 ds + 2\int_0^t \left(\mu_m^{(k)}(s)\right)^2 ||\frac{\partial}{\partial x}\left(A\frac{\partial u_m^{(k)}}{\partial x}(s)\right)||^2 ds \\ &= I_5^{(1)} + I_5^{(2)}. \end{aligned} \tag{3.41}$$

We shall estimate step by step two integrals $I_5^{(1)}$, $I_5^{(2)}$.

Estimate $I_5^{(1)}$: Using the inequalities (3.21) and $\left(\sum_{i=0}^{N-1} a_i\right)^2 \leq N\sum_{i=0}^{N-1} a_i^2$, for all $a_0, a_1, ..., a_{N-1} \in \mathbb{R}$, it follows from (3.23), that

$$\begin{aligned} I_5^{(1)} &= 2\int_0^t ||F_m^{(k)}(s)||^2 ds \leq 2N\widehat{K}_{N-1}^2 \sum_{i=0}^{N-1} \widetilde{a}_i^2 \int_0^t \left(s_m^{(k)}(s)\right)^i ds \\ &\leq 2N\widehat{K}_{N-1}^2 \sum_{i=0}^{N-1} \widetilde{a}_i^2 \int_0^t \left[1 + \left(s_m^{(k)}(s)\right)^{N_0}\right] ds \\ &\leq 2N\widehat{K}_{N-1}^2 \sum_{i=0}^{N-1} \widetilde{a}_i^2 \left[T + \int_0^t \left(s_m^{(k)}(s)\right)^{N_0} ds\right]. \end{aligned} \tag{3.42}$$

Estimate $I_5^{(2)}$: By using the assumption (H_4, (i)), we deduce from $(3.10)_1$, (3.15), (3.16), that

$$\begin{aligned} |\mu_m^{(k)}(t)| &\leq \mu_0 \left(1 + ||u_m^{(k)}(t)||^{2p} + ||\bigtriangledown u_m^{(k)}(t)||^{2p}\right) \\ &\leq \mu_0 \left[1 + 2\left(a_0\mu_*\right)^{-p}\left(s_m^{(k)}(t)\right)^p\right]. \end{aligned} \tag{3.43}$$

Hence, we obtain from (3.21), (3.43), that

$$\begin{aligned} I_5^{(2)} &= 2\int_0^t \left(\mu_m^{(k)}(s)\right)^2 ||\frac{\partial}{\partial x}\left(A\frac{\partial u_m^{(k)}}{\partial x}(s)\right)||^2 ds \\ &\leq \frac{2\mu_0^2}{\mu_*}\int_0^t \left[1 + 2\left(a_0\mu_*\right)^{-p}\left(s_m^{(k)}(s)\right)^p\right]^2 s_m^{(k)}(s) ds \\ &\leq \frac{4\mu_0^2}{\mu_*}\left[1 + 4\left(a_0\mu_*\right)^{-2p}\right]^2 \int_0^t \left[1 + \left(s_m^{(k)}(s)\right)^{2p}\right] s_m^{(k)}(s) ds \\ &\leq \frac{8\mu_0^2}{\mu_*}\left[1 + 4\left(a_0\mu_*\right)^{-2p}\right]^2 \int_0^t \left[1 + \left(s_m^{(k)}(s)\right)^{N_0}\right] ds \\ &\leq \frac{8\mu_0^2}{\mu_*}\left[1 + 4\left(a_0\mu_*\right)^{-2p}\right]^2 \left[T + \int_0^t \left(s_m^{(k)}(s)\right)^{N_0}\right] ds \\ &= \widetilde{\mu}_5\left[T + \int_0^t \left(s_m^{(k)}(s)\right)^{N_0}\right] ds, \end{aligned} \tag{3.44}$$

where

$$\widetilde{\mu}_5 = \frac{8\mu_0^2}{\mu_*}\left[1 + 4\left(a_0\mu_*\right)^{-2p}\right]^2. \tag{3.45}$$

It follows from (3.41), (3.42), (3.44), that

$$I_5 \leq K_N^{(1)}\left[T + \int_0^t \left(s_m^{(k)}(s)\right)^{N_0} ds\right], \tag{3.46}$$

where

$$K_N^{(1)} = \widetilde{\mu}_5 + 2N\widehat{K}_{N-1}^2 \sum_{i=0}^{N-1} \widetilde{a}_i^2. \tag{3.47}$$

Now, we need an estimate on the term $S_m^{(k)}(0)$. We have

$$\begin{aligned} S_m^{(k)}(0) &= ||\widetilde{u}_{1k}||^2 + ||\widetilde{u}_{1k}||_a^2 \\ &\quad + \mu\left(0, ||\widetilde{u}_{0k}||^2, ||\bigtriangledown \widetilde{u}_{0k}||^2\right)\left[||\widetilde{u}_{0k}||_a^2 + ||\frac{\partial}{\partial x}\left(A\frac{\partial \widetilde{u}_{0k}}{\partial x}\right)||^2\right]. \end{aligned} \tag{3.48}$$

By means of the convergences (3.9) we can deduce the existence of a constant $M > 0$ independent of k and m such that

$$2S_m^{(k)}(0) + 4A(1)\left|f_1(0)\nabla\widetilde{u}_{0k}(1)\right| + 8A^2(1)\widetilde{\mu}_4 f_1^2(0) \leq \frac{1}{2}M^2. \tag{3.49}$$

Finally, it follows from (3.11)-(3.13), (3.22), (3.30), (3.33), (3.39), (3.46), (3.49), with $\beta = \frac{1}{4}$, that

$$s_m^{(k)}(t) \leq \frac{1}{2}M^2 + T\widetilde{D}_2(M,T) + \widetilde{D}_1(M,T)\int_0^t \left(s_m^{(k)}(s)\right)^{N_0} ds, \tag{3.50}$$

for $0 \leq t \leq T_m^{(k)} \leq T$, where

$$\begin{cases} \widetilde{D}_1(M,T) = 4A(1)\sqrt{\widetilde{\mu}_4}||f_1'||_{L^\infty}^2 + 2K_N^{(1)} + \frac{6\widetilde{\mu}_1}{\mu_*} \\ \qquad\qquad + 2\left[2\widehat{K}_{N-1} + \sqrt{2h + A_{\max}}\left(\widehat{K}_{N-1} + \widetilde{K}_{N-1}\right)\right]\sum_{i=0}^{N-1}\widetilde{a}_i, \\ \widetilde{D}_2(M,T) = \widetilde{D}_1(M,T) + 8A^2(1)\widetilde{\mu}_4 T||f_1'||_{L^\infty}^2. \end{cases} \tag{3.51}$$

Then, we have the following lemma.

Lemma 3.3. *There exists a constant $T > 0$ independent of k and m such that*

$$s_m^{(k)}(t) \leq M^2 \ \forall t \in [0,T], \text{ for all } k \text{ and } m. \tag{3.52}$$

Proof. Put

$$Y(t) = \frac{1}{2}M^2 + T\widetilde{D}_2(M,T) + \widetilde{D}_1(M,T)\int_0^t \left(s_m^{(k)}(s)\right)^{N_0} ds, \ 0 \leq t \leq T. \tag{3.53}$$

Clearly

$$\begin{cases} Y(t) > 0,\ 0 \le s_m^{(k)}(t) \le Y(t),\ 0 \le t \le T, \\ Y'(t) \le \widetilde{D}_1(M,T) Y^{N_0}(t),\ 0 \le t \le T, \\ Y(0) = \frac{1}{2}M^2 + T\widetilde{D}_2(M,T). \end{cases} \tag{3.54}$$

Put $Z(t) = Y^{1-N_0}(t)$, after integrating of (3.54)

$$\begin{aligned} Z(t) &\ge \left(\frac{1}{2}M^2 + T\widetilde{D}_2(M,T)\right)^{1-N_0} - (N_0-1)\,\widetilde{D}_1(M,T)t \\ &\ge \left(\frac{1}{2}M^2 + T\widetilde{D}_2(M,T)\right)^{1-N_0} - (N_0-1)\,\widetilde{D}_1(M,T)T,\ \forall t \in [0,T]. \end{aligned} \tag{3.55}$$

Notice that, from (3.51), we have

$$\begin{aligned} &\lim_{T\to 0^+} \left[\left(\frac{1}{2}M^2 + T\widetilde{D}_2(M,T)\right)^{1-N_0} - (N_0-1)\,\widetilde{D}_1(M,T)T\right] \\ &= \left(\frac{1}{2}M^2\right)^{1-N_0} > \left(M^2\right)^{1-N_0}. \end{aligned} \tag{3.56}$$

Then, from (3.56), we can always choose the constant $T > 0$ such that

$$\left(\tfrac{1}{2}M^2 + T\widetilde{D}_2(M,T)\right)^{1-N_0} - (N_0-1)\,\widetilde{D}_1(M,T)T > \left(M^2\right)^{1-N_0}. \tag{3.57}$$

Finally, it follows from (3.54), (3.55) and (3.57), that

$$0 \le s_m^{(k)}(t) \le Y(t) = \frac{1}{\sqrt[N_0-1]{Z(t)}} \le M^2,\ \forall t \in [0,T]. \tag{3.58}$$

The proof of Lemma 3.3 is complete. □

Remark 3.1. The function

$$S(t) = \left[\left(\frac{1}{2}M^2 + T\widetilde{D}_2(M,T)\right)^{1-N_0} - (N_0-1)\,\widetilde{D}_1(M,T)t\right]^{\frac{1}{1-N_0}},\ 0 \le t \le T,$$

is the maximal solution of the following Volterra integral equation with non-decreasing kernel (see [6]).

$$S(t) = \tfrac{1}{2}M^2 + T\widetilde{D}_2(M,T) + \widetilde{D}_1(M,T)\int_0^t S^{N_0}(s)ds,\ 0 \le t \le T. \tag{3.59}$$

By Lemma 3.3, we can take constant $T_m^{(k)} = T$ for all m and k. Therefore, we have

$$u_m^{(k)} \in W(M,T) \text{ for all } m \text{ and } k. \tag{3.60}$$

From (3.60) we can extract from $\{u_m^{(k)}\}$ a subsequence $\{u_m^{(k_j)}\}$ such that

$$\left\{\begin{array}{llll} u_m^{(k_j)} \to u_m & \text{in} & L^\infty(0,T;V\cap H^2) & \text{weak*ly}, \\ \dot{u}_m^{(k_j)} \to \dot{u}_m & \text{in} & L^\infty(0,T;V) & \text{weak*ly}, \\ \ddot{u}_m^{(k_j)} \to \ddot{u}_m & \text{in} & L^2(Q_T) & \text{weakly}, \end{array}\right. \tag{3.61}$$

$$u_m \in W(M,T). \tag{3.62}$$

We can easily check from (3.8) – (3.10), (3.61), (3.62) that u_m satisfies (3.4) – (3.6) in $L^2(0,T)$. On the other hand, it follows from $(3.4)_1$ and $u_m \in W(M,T)$ that $\ddot{u}_m = \mu_m(t)\frac{\partial}{\partial x}(Au_{mx}) + F_m \in L^\infty(0,T;L^2)$, hence $u_m \in W_1(M,T)$ and the proof of Theorem 3.1 is complete. □

Theorem 3.4. *Let (H_1) -(H_5) hold. Then*

(i) *There exist constants $M > 0$ and $T > 0$ satisfying (3.49), (3.57) such that the problem (1.1) – (1.3) has a local unique weak solution $u \in W_1(M,T)$.*

(ii) *The recurrent sequence $\{u_m\}$ defined by (3.4) – (3.6), converges at a rate of order N to the solution u strongly in the space $W_1(T) = \{v \in L^\infty(0,T;V) : \dot{v} \in L^\infty(0,T;L^2)\}$ in the sense*

$$\begin{aligned} &||u_m - u||_{L^\infty(0,T;V)} + ||\dot{u}_m - \dot{u}||_{L^\infty(0,T;L^2)} \\ &\le C\left(||u_{m-1} - u||_{L^\infty(0,T;V)} + ||\dot{u}_{m-1} - \dot{u}||_{L^\infty(0,T;L^2)}\right)^N, \end{aligned}$$

for all $m \ge 1$, where C is a suitable constant.

Furthermore, we have also the estimation

$$||u_m - u||_{L^\infty(0,T;V)} + ||\dot{u}_m - \dot{u}||_{L^\infty(0,T;L^2)} \le C_T\,(k_T)^{N^m}, \tag{3.63}$$

for all $m \ge 1$, where C_T and $k_T < 1$ are positive constants depending only on T.

Proof. First, we note that $W_1(T)$ is a Banach space with respect to the norm (see [7]):

$$||v||_{W_1(T)} = ||v||_{L^\infty(0,T;V)} + ||\dot{v}||_{L^\infty(0,T;L^2)}.$$

We shall prove that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Let $v_m = u_{m+1} - u_m$. Then v_m satisfies the variational problem

$$\left\{\begin{array}{l} \langle \ddot{v}_m(t), v\rangle + \mu_{m+1}(t)a(v_m(t),v) = (\mu_{m+1}(t) - \mu_m(t))\,\langle \frac{\partial}{\partial x}(Au_{mx}(t)), v\rangle \\ \qquad\qquad + \langle F_{m+1}(t) - F_m(t), v\rangle \;\forall v \in V, \\ v_m(0) = \dot{v}_m(0) = 0, \end{array}\right. \tag{3.64}$$

where

$$\left\{\begin{array}{l} \mu_m(t) = \mu\left(t, ||u_m(t)||^2, ||u_{mx}(t)||^2\right), \\ F_m(x,t) = \sum_{i=0}^{N-1} \frac{1}{i!} D_3^i f(x,t,u_{m-1})\,(u_m - u_{m-1})^i. \end{array}\right. \tag{3.65}$$

Taking $w = \dot{v}_m$ in $(3.64)_1$, after integrating in t we get

$$\begin{aligned}\sigma_m(t) &= \int_0^t \dot{\mu}_{m+1}(s)||v_m(s)||_a^2 ds \\ &+2\int_0^t (\mu_{m+1}(s)-\mu_m(s)) \left\langle \tfrac{\partial}{\partial x}\left(A\tfrac{\partial u_m}{\partial x}(s)\right), \dot{v}_m(s)\right\rangle ds \\ &+2\int_0^t \left\langle F_{m+1}(s)-F_m(s), \dot{v}_m(s)\right\rangle ds = \sum_{k=1}^3 J_k,\end{aligned} \tag{3.66}$$

where

$$\begin{aligned}\sigma_m(t) &= ||\dot{v}_m(t)||^2 + \mu_{m+1}(t)a(v_m(t),v_m(t)) \\ &\geq ||\dot{v}_m(t)||^2 + \mu_*||v_m(t)||_a^2 \equiv E_m(t).\end{aligned} \tag{3.67}$$

We shall estimate step by step all integrals J_k, $k = 1, 2, 3$.

First, by using the assumption (H_4, (ii), (iii), (iv)), we deduce from (3.3), (3.62), that

$$\begin{aligned}&\left|\dot{\mu}_{m+1}(t)\right| \\ &\leq \mu_1\left(1+||u_{m+1}(t)||^{2p}+||\nabla u_{m+1}(t)||^{2p}\right) \\ &\quad + 2\mu_2\left(1+||u_{m+1}(t)||^{2p-2}+||\nabla u_{m+1}(t)||^{2p}\right)||u_{m+1}(t)||||\dot{u}_{m+1}(t)|| \\ &\quad + 2\mu_3\left(1+||u_{m+1}(t)||^{2p}+||\nabla u_{m+1}(t)||^{2p-2}\right)||\nabla u_{m+1}(t)||||\nabla \dot{u}_{m+1}(t)|| \\ &\leq \mu_1\left[1+2\left(\frac{1}{\sqrt{a_0}}||u_{m+1}(t)||_a\right)^{2p}\right] \\ &\quad + 2\mu_2\left[1+\left(\frac{1}{\sqrt{a_0}}||u_{m+1}(t)||_a\right)^{2p-2}+\left(\frac{1}{\sqrt{a_0}}||u_{m+1}(t)||_a\right)^{2p}\right]\times \\ &\quad \times\frac{1}{\sqrt{a_0}}||u_{m+1}(t)||_a||\dot{u}_{m+1}(t)|| \\ &\quad + 2\mu_3\left(1+\left(\frac{1}{\sqrt{a_0}}||u_{m+1}(t)||_a\right)^{2p}+\left(\frac{1}{\sqrt{a_0}}||u_{m+1}(t)||_a\right)^{2p-2}\right)\times \\ &\quad \times\frac{1}{\sqrt{a_0}}||u_{m+1}(t)||_a||\nabla \dot{u}_{m+1}(t)|| \\ &\leq \mu_1\left[1+2\left(\frac{M}{\sqrt{a_0}}\right)^{2p}\right]+2(\mu_2+\mu_3)\left[1+\left(\frac{M}{\sqrt{a_0}}\right)^{2p-2}+\left(\frac{M}{\sqrt{a_0}}\right)^{2p}\right]\frac{M^2}{\sqrt{a_0}} \\ &\equiv \widetilde{M}_1,\end{aligned} \tag{3.68}$$

$$\begin{aligned}|\mu_{m+1}(t)-\mu_m(t)| &\leq 2\left(1+M^{2p-2}+M^{2p}\right)M\left[\mu_2\|v_m(t)\|+\mu_3\|\nabla v_m(t)\|\right] \\ &\leq 2\left(1+M^{2p-2}+M^{2p}\right)M\left[\mu_2\|v_m(t)\|_a+\mu_3\|v_m(t)\|_a\right]\end{aligned}$$

$$= 2\left(1 + M^{2p-2} + M^{2p}\right) M \left(\mu_2 + \mu_3\right) \left\|v_m(t)\right\|_a \equiv \widetilde{M}_2 \left\|v_m(t)\right\|_a, \tag{3.69}$$

$$\begin{aligned} ||\frac{\partial}{\partial x}\left(A\frac{\partial u_m}{\partial x}(t)\right)|| &\leq ||A\Delta u_m(t)|| + ||\nabla A \nabla u_m(t)|| \leq \widetilde{M}_3 \\ &\leq A_{\max}||\Delta u_m||_{L^\infty(0,T:L^2)} + ||\nabla A||_{L^\infty(\Omega)}||\nabla u_m||_{L^\infty(0,T:L^2)} \\ &\leq \left(A_{\max} + ||\nabla A||_{L^\infty(\Omega)}\right) M \equiv \widetilde{M}_3. \end{aligned} \tag{3.70}$$

Hence, it follows from (3.62), (3.67)–(3.70), that

$$\begin{aligned} |J_1| &= \left|\int_0^t \dot{\mu}_{m+1}(s) \left\|v_m(s)\right\|_a^2 ds\right| \\ &\leq \int_0^t \left|\dot{\mu}_{m+1}(s)\right| ||u_m(s)||_a^2 ds \leq \frac{\widetilde{M}_1}{\mu_*}\int_0^t E_m(s)ds, \end{aligned} \tag{3.71}$$

$$\begin{aligned} |J_2| &= 2\left|\int_0^t \left(\mu_{m+1}(s) - \mu_m(s)\right)\left\langle \frac{\partial}{\partial x}\left(A\frac{\partial u_m}{\partial x}(s)\right), \dot{v}_m(s)\right\rangle ds\right| \\ &\leq 2\int_0^t \left|\mu_{m+1}(s) - \mu_m(s)\right| ||\frac{\partial}{\partial x}\left(A\frac{\partial u_m}{\partial x}(s)\right)|| ||\dot{v}_m(s)||ds \\ &\leq 2\widetilde{M}_2\widetilde{M}_3\frac{1}{\sqrt{\mu_*}}\int_0^t E_m(s)ds. \end{aligned} \tag{3.72}$$

On the other hand, by using Taylor's expansion of the function $f(x,t,u_m)$ around the point u_{m-1} up to order N, we obtain

$$\begin{aligned} &f(x,t,u_m) - f(x,t,u_{m-1}) \\ &= \sum_{i=1}^{N-1}\frac{1}{i!}D_3^i f(x,t,u_{m-1})\left(v_{m-1}\right)^i + \frac{1}{N!}D_3^N f(x,t,\lambda_m)\left(v_{m-1}\right)^N, \end{aligned} \tag{3.73}$$

where $\lambda_m = \lambda_m(x,t) = u_{m-1} + \theta_1\left(u_m - u_{m-1}\right),\ 0 < \theta_1 < 1$.

Hence, it follows from (3.6), (3.73), that

$$\begin{aligned} &F_{m+1}(x,t) - F_m(x,t) \\ &= \sum_{i=1}^{N-1}\frac{1}{i!}D_3^i f(x,t,u_m)\left(v_m\right)^i + \frac{1}{N!}D_3^N f(x,t,\lambda_m)\left(v_{m-1}\right)^N. \end{aligned} \tag{3.74}$$

Then we deduce, from (3.62), (3.67) and (3.74), that

$$\begin{aligned} &||F_{m+1}(t) - F_m(t)|| \\ &\leq \sum_{i=1}^{N-1}\frac{K_i}{i!}\left(\frac{1}{\sqrt{a_0}}||v_m(t)||_a\right)^i + \frac{K_N}{N!}\left(\frac{1}{\sqrt{a_0}}||v_{m-1}(t)||_a\right)^N \\ &\leq \sum_{i=1}^{N-1}\frac{K_i}{i!}\left(\frac{1}{\sqrt{a_0}}\right)^i ||v_m(t)||_a^{i-1}||v_m(t)||_a + \frac{K_N}{N!}\left(\frac{1}{\sqrt{a_0}}\right)^N ||v_{m-1}(t)||_a^N \end{aligned}$$

$$\leq \sum_{i=1}^{N-1} \frac{K_i}{i!} \left(\frac{1}{\sqrt{a_0}}\right)^i \frac{M^{i-1}}{\sqrt{\mu_*}} \sqrt{E_m(t)} + \frac{K_N}{N!} \left(\frac{1}{\sqrt{a_0}}\right)^N \frac{1}{\left(\sqrt{\mu_*}\right)^N} \left(\sqrt{E_{m-1}(t)}\right)^N$$

(3.75)

$$= \rho_T^{(1)} \sqrt{E_m(t)} + \rho_T^{(2)} \left(\sqrt{E_{m-1}(t)}\right)^N,$$

where

(3.76) $$\rho_T^{(1)} = \sum_{i=1}^{N-1} \frac{K_i}{i!} \left(\frac{1}{\sqrt{a_0}}\right)^i \frac{M^{i-1}}{\sqrt{\mu_*}}, \ \rho_T^{(2)} = \frac{K_N}{N!} \frac{1}{\left(\sqrt{a_0 \mu_*}\right)^{N+1}}.$$

Then we deduce, from (3.67) and (3.75), that

$$\begin{aligned} J_3 &= 2 \int_0^t \langle F_{m+1}(s) - F_m(s), \dot{v}_m(s) \rangle ds \\ &\leq 2 \int_0^t ||F_{m+1}(s) - F_m(s)|| ||\dot{v}_m(s)|| ds \\ &\leq 2 \int_0^t \left[\rho_T^{(1)} \sqrt{E_m(s)} + \rho_T^{(2)} \left(\sqrt{E_{m-1}(s)}\right)^N \right] \sqrt{E_m(s)} ds \end{aligned}$$

(3.77) $$\leq \left(2\rho_T^{(1)} + \rho_T^{(2)}\right) \int_0^t E_m(s) ds + \rho_T^{(2)} \int_0^T E_{m-1}^N(s) ds.$$

Combining (3.66), (3.67), (3.71), (3.72) and (3.77), we then have

(3.78) $$E_m(t) \leq \rho_T^{(2)} \int_0^T E_{m-1}^N(s) ds + \rho_T^{(3)} \int_0^t E_m(s) ds,$$

where

(3.79) $$\rho_T^{(3)} = \frac{\widetilde{M}_1}{\mu_*} + \frac{2\widetilde{M}_2 \widetilde{M}_3}{\sqrt{\mu_*}} + 2\rho_T^{(1)} + \rho_T^{(2)}.$$

By using Gronwall's lemma, we obtain from (3.78) that

(3.80) $$||v_m||_{W_1(T)} \leq \mu_T ||v_{m-1}||_{W_1(T)}^N,$$

where μ_T is the constant given by

(3.81) $$\mu_T = \left(1 + \frac{1}{\sqrt{\mu_*}}\right) \sqrt{T \rho_T^{(2)} (1 + \mu_*)^N \exp(T \rho_T^{(3)})}.$$

Hence, we obtain from (3.78) that

(3.82) $$||u_m - u_{m+p}||_{W_1(T)} \leq (1 - k_T)^{-1} (\mu_T)^{\frac{-1}{N-1}} (k_T)^{N^m},$$

for all m and p where $k_T = 2M (\mu_T)^{\frac{1}{N-1}} < 1$. It follows that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $u \in W_1(T)$ such that $u_m \to u$ strongly in $W_1(T)$. Thus, by applying a similar argument used in the proof of Theorem 3.1, $u \in W_1(M, T)$ is the local unique weak solution of problem (1.1)-(1.3). Passing to the limit as $p \to +\infty$ for fixed m, we obtain the estimate (3.63) from (3.82). This completes the proof of Theorem 3.4. □

Remark 3.2. In order to construct a N-order iterative scheme, we need the condition $f \in C^N([0,1] \times \mathbb{R}_+ \times \mathbb{R})$. Then, we get a convergent sequence at a rate of order N to a local unique weak solution of problem and the existence follows. However, the above condition of f can be relaxed if we only consider the existence of solution, see [9]–[12].

Acknowledgements

The authors wish to express their sincere thanks to the referee for his/her valuable comments and important remarks. The comments uncovered several weaknesses in the presentation of the paper and helped us to clarify it.

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