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CYCLIC MODULES OVER SIMPLE GOLDIE RINGS

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Abstract. In [10, Theorem A] it was shown that if every cyclic singular right module over a simple ring R is CS, then R is right noetherian. In this note we extend this result to cyclic modules over a simple right Goldie ring, and apply it to characterize simple noetherian rings and simple SI rings by using a single nonzero principal one-sided ideal of the ring.

1. Introduction

All rings are associative with identity, and all modules are unitary modules. For a module M (over a ring R) we consider the following conditions.

(C_1) Every submodule of M is essential in a direct summand of M,

(C_2) Every submodule isomorphic to a direct summand of M is itself a direct summand of M, and

(C_3) For any direct summands $A,\ B \subseteq M$ with $A \cap B = 0$, $A \oplus B$ is also a direct summand of M.

A module is called a CS (or extending) module if it satisfies (C_1). A ring R is defined to be a right CS ring if the module R_R is CS.

If M satisfies (C_1) and (C_2), then M is said to be a continuous module.

M is defined to be quasi-continuous, if it satisfies (C_1) and (C_3).

A ring R is said to be right (quasi-) continuous if R_R is (quasi)-continuous. For the basic properties of CS-modules and (quasi)-continuous rings and modules we refer to the books [3] and [13], respectively.

The composition length of a module M is denoted by $l(M)$; $\mathrm{Soc}(M)$ denotes the socle of M. Let n be a positive integer, then the direct sum of n copies of a module M is denoted by M^n. For a module M, by $\sigma[M]$ we denote the full subcategory of Mod-R whose objects are submodules of M-generated modules (see Wisbauer [15]).

For the general backgrounds of modules and rings we refer to [1], [4], [12], and [15].

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By [10, Theorem A], if every cyclic singular right module over a simple ring R is CS, then R is right noetherian. Note that this theorem is not correct for non-simple rings. In this note we extend this result to cyclic modules over a simple right Goldie ring, and apply it to characterize simple noetherian rings and simple SI rings by using a single nonzero principal one-sided ideal of the ring. For obtaining our results we develop techniques presented in [10] and [11].

2. Results

First we list some lemmas that are important in our proofs. The first lemma is due to B. Osofsky and P.F. Smith [14], the second is a result of J.T. Stafford (see [1, Theorem 14.1] and also [10, Lemma 3.1]).

Lemma 2.1. *Let M be a cyclic module. If every cyclic submodule in $\sigma[M]$ is CS, then M has finite uniform dimension.*

Lemma 2.2. *Let M be a singular module over a simple right Goldie ring R that is not artinian. If $M = aR \oplus bR$, a, $b \in M$ such that bR has finite composition length, then $M = (a + bx)R$ for some $x \in R$, in particular, M is cyclic.*

The following lemma is taken from [9]:

Lemma 2.3. *Let M be a module such that every factor module of M has finite uniform dimension. If every simple module in $\sigma[M]$ is M-injective, then M is noetherian.*

Let N_R be a module. A module X_R is defined to be N-singular if there is a module $A \in \sigma[N]$ containing an essential submodule E such that $X \cong A/E$. The class of N-singular modules is closed under submodules, factors, direct sums and essential extensions. Hence if X is N-singular, then every module in $\sigma[X]$ is also N-singular. For $N = R$ we obtain the usual concept of the singularity of modules in Mod-R.

We are now able to prove the following:

Theorem 2.4. *Let M be a cyclic right module over a simple right Goldie ring R. If every cyclic M-singular module in $\sigma[M]$ is CS, then $M/\mathrm{Soc}(M)$ is noetherian.*

Proof. Assume that M is a cyclic right module over a simple right Goldie ring R. If $\mathrm{Soc}(R_R) \neq 0$, then as R is simple, $R = \mathrm{Soc}(R_R)$, i.e. R is a simple artinian ring. In this case M is noetherian and artinian. Hence, throughout the proof, we assume that $\mathrm{Soc}(R_R) = 0$.

Let $E \subseteq M$ be an essential submodule. Then $N = M/E$ is a cyclic M-singular module in $\sigma[M]$. Being M-singular, every cyclic module in $\sigma[N] \subseteq \sigma[M]$ is CS. By Lemma 2.1, N has finite uniform dimension. Let α be an ordinal. We define the socle series of N as follows:

$$S_1 = \mathrm{Soc}(N), \quad S_\alpha / S_{\alpha-1} = \mathrm{Soc}(N/N_{\alpha-1}),$$

and

$$S_\alpha = \cup_{\beta<\alpha} S_\beta$$

if α is a limit ordinal. Then for the submodule $S = \cup_\alpha S_\alpha$, $V = N/S$ has zero socle.

Since V is CS and has finite uniform dimension (by Lemma 2.1), V is a direct sum of finitely many uniform submodules. Hence we may assume (without loss of generality) that V is a uniform module. Let U be a simple M-singular module in $\sigma[M]$, then by Lemma 2.2, $T = V \oplus U$ is a cyclic M-singular module in $\sigma[M]$. Hence T is CS. It is clear that $\mathrm{Soc}(T) = U$. Next we show that U is V-injective. Let A be an arbitrary nonzero submodule of V and let $f:\ A \to U$ be a homomorphism. Define $B = \{a - f(a) |\ a \in A\}$. Then B is contained essentially in a direct summand $B^* \subseteq T$, i.e. we have $T = B^* \oplus C$ for some submodule $C \subseteq T$. But $B^* \cap U = 0$ and $\mathrm{Soc}(T) = U$, a fully invariant submodule, we must have $U \subseteq C$. Since U is closed in T, and C is uniform (because, the uniform dimension of T is 2), we have $U = C$. Thus,

$$T = B^* \oplus U.$$

From this decomposition, let π be the projection of T onto U. Then we can check that the mapping $\pi' = (\pi|_V)$ is an extension of f from V to U. We conclude that every simple module in $\sigma[V]$ is V-injective. On the other hand, using Lemma 2.1, we see that every factor module of V has finite uniform dimension. Thus, by Lemma 2.3, the module V is noetherian.

As $V = N/S$, to show that N is noetherian we have to show that S_R is noetherian, or equivalently, that S_R has finite composition length. Since S_1 and each $S_{\alpha+1}/S_\alpha$ have finite composition lengths, it is enough to show that $S = S_2$. If $S_2 \neq S_3$ there is an $y \in S_3$ with $yR \nsubseteq S_2$. We can choose y so that $(yR+S_2)/S_2$ is a simple module. Since yR is CS, we have

$$yR = H_1 \oplus \cdots \oplus H_m,$$

where each H_i is uniform. By the choice of y, there is some H_i, say H_1 with $H_1 \nsubseteq S_2$. Again since $H_1/\mathrm{Soc}(K_1)$ is CS, there are finitely many submodules $K_1, \cdots, K_t$ of H_1 such that

$$H_1/\mathrm{Soc}(H_1) = (K_1/\mathrm{Soc}(H_1)) \oplus \cdots \oplus (K_t/\mathrm{Soc}(H_1)),$$

where each $K_j/\mathrm{Soc}(H_1)$ is simple or uniform with $l[K_j/\mathrm{Soc}(H_1)] = 2$. It is sure that there is some K_j, say K_1 with $l[K_1/\mathrm{Soc}(H_1)] = 2$. Since H_1 is uniform, it follows that K_1 is then a uniserial module with the unique composition series $\mathrm{Soc}(H_1) \subset K \subset K_1$. Notice that K_1 is cyclic, hence by Lemma 2.2, $K_1 \oplus (H/\mathrm{Soc}(H_1))$ is cyclic, moreover it is M-singular. Hence $K_1 \oplus (H/\mathrm{Soc}(H_1))$ is CS by our hypothesis. But this is a contradiction to a result by Osofsky (see [3, Corollary 7.4]) that this module cannot be CS. This contradiction means we must have $S_2 = S_3$, thus $S = S_2$ which has finite composition length.

So far we have shown that M is noetherian modulo each of its essential submodules. Hence by [3, 5.15], $M/\mathrm{Soc}(M)$ is noetherian. $\square$

The singular submodule of a right R-module M is denoted by $Z(M)$, i.e., $Z(M)$ is the set of those elements $x \in M$ such that the right annihilator $r_R(x)$

of x in R is an essential right ideal of R. As a consequence of Theorem 2.4 we obtain the following result.

Remark. As observed in [2, Lemma 2.1], if R is a simple ring and $A \subseteq R$ is a nonzero right ideal, then there are elements $a_1, \cdots, a_n \in A$ such that $R_R = a_1A + \cdots + a_nA$ for some positive integer n. It follows that R_R is a direct summand of A^n.

Corollary 2.5. *For a simple ring R the following conditions are equivalent:*

(i) *Every cyclic singular right R-module is CS,*

(ii) *There exists a cyclic right R-module X with $X \neq Z(X)$ such that every cyclic X-singular module in $\sigma[X]$ is CS.*

In this case, R is right noetherian.

Proof. (i)$\Rightarrow$(ii) is clear. Now assume that (ii) holds. For $\mathrm{Soc}(R_R) \neq 0$ the statement is clear. Hence we assume that $\mathrm{Soc}(R_R) = 0$. There is $x \in X$ such that $X = xR$. Since $X \neq Z(X)$ the annihilator $r_R(x)$ in R is not an essential right ideal of R. As $X = xR \cong R_R/r_R(x)$, X contains a nonzero cyclic submodule Y that is isomorphic to a principal right ideal of R. Hence Y is nonsingular and $\mathrm{Soc}(Y_R) = 0$. It is clear that $\sigma[Y] \subseteq \sigma[X]$. Using the above remark we see that R_R is isomorphic to a direct summand of Y^k for some positive integer k. Hence R_R is isomorphic to an object from $\sigma[Y] \subseteq \sigma[X]$. It follows that $\sigma[X] = \text{Mod-}R$. Hence (ii) $\Rightarrow$ (i).

In case of (ii) we have $\mathrm{Soc}(Y) = 0$. Applying condition (ii) for $\sigma[Y/E]$ for each essential submodule $E \subseteq Y$we see that $(Y/E)_R$ has finite uniform dimension by Lemma 2.1. Hence, by [3. 5.14], $Y/Soc(Y)$ $(= Y)$ has finite uniform dimension. In particular R has a uniform right ideal. Hence R is right Goldie by Hart [7]. Now we can apply Theorem 2.4 to see that Y_R is noetherian. Thus, as a direct summand of Y^k, the ring R is right noetherian. □

Theorem 2.6. *Let R be a simple right Goldie ring, and Y be a (nonzero) cyclic right R-module. If every cyclic Y-singular module in $\sigma[Y]$ is quasi-continuous, then Y/E is semisimple for any essential submodule $E \subseteq Y$.*

Proof. If $\mathrm{Soc}(R_R) \neq 0$, then R is a semisimple artinian ring, and hence the statement is obvious. We consider the case that $\mathrm{Soc}(R_R) = 0$.

Assume that Y is a nonzero cyclic right R-module such that every cyclic Y-singular module in $\sigma[Y]$ is quasi-continuous. By Theorem 2.4, $Y/\mathrm{Soc}(Y)$ is right noetherian. We aim to show that

(*) for every essential submodule $E \subseteq Y$, Y/E is semisimple.

First consider the case that $X = Y/E$ is artinian. As $\mathrm{Soc}(X_R)$ has finite length, using Lemma 2.2 we can inductively show that $X \oplus \mathrm{Soc}(X)$ is cyclic. Since $X \oplus \mathrm{Soc}(X) \in \sigma[Y]$ and Y-singular it is quasi-continuous by our hypothesis. Hence $\mathrm{Soc}(X)$ is X-injective, and so $\mathrm{Soc}(X)$ splits in X. This shows that $X = \mathrm{Soc}(X)$, i.e., X is semisimple whenever it is artinian. Thus to prove (*) we need to show that Y/E is artinian for any essential submodule $E \subseteq Y$.

Assume on the contrary that for some essential submodule $E \subseteq Y$, Y/E is *not* artinian. As Y_R is noetherian modulo its socle, there is an essential submodule $F \subseteq Y$ which is maximal with respect to the condition that $V = Y/F$ is not artinian. If V is not uniform then there are nonzero submodules V_1, $V_2 \subseteq V$ with $V_1 \cap V_2 = 0$. Let U_i, $(i = 1, 2)$ be the preimage of V_i in Y with respect to the canonical homomorphism $Y \to Y/F$ $(= V)$. Then by the maximality of F, Y/U_i is artinian. It follows that V $(= Y/F)$ is artinian, a contradiction. Thus V must be uniform. Moreover, by the same reason and by the choice of F we can show that $\mathrm{Soc}(V) = 0$.

Also by the choice of F, for any nonzero submodule $T \subseteq V$, V/T is artinian, hence semisimple. Therefore there exist submodules T and U of V with $0 \neq T \subset U \subset V$ such that U/T is a direct sum of finitely many simple modules. Consider the module $Q = V \oplus U$. Since V is cyclic and $Q/(0, T) \cong V \oplus (U/T)$ we can use Lemma 2.2 to see that $Q/(0, T)$ is cyclic. Let $x \in Q$ so that the coset $x + (0, T)$ generates $Q/(0, T)$, i.e., $[xR+(0, T)]/(0, T) = Q/(0, T)$. Obviously we can choose x so that xR contains $(V, 0)$. Hence $xR = V \oplus W$ where $(0, W) = xR \cap (0, U)$. Since xR is quasi-continuous, W is V-injective. As xR is not uniform, $W \neq 0$. Thus U contains a nonzero submodule that is V-injective, and so that submodule must split in V. This is a contradiction to the fact that V is uniform. Hence for any essential submodule E of Y, the factor module Y/E is artinian. This proves (*). □

Remark. It is still unknown if Theorems 2.4 and 2.6 hold without the assumption that R is right Goldie.

SI-rings, i.e., rings over which every singular module is injective, were introduced and studied by Goodearl in [6]. The concept of SI-modules were defined and investigated in [8]. A module M is called SI if every M-singular module in $\sigma[M]$ is M-injective. In [11] it was shown that a simple ring R is right SI if and only if every cyclic singular right R-module is quasi-continuous. Also this theorem doesn't hold for non-simple rings. We obtain the same result by considering only one single principal right ideal of a simple ring as a consequence of the result bellow.

For detailed discussions on SI-rings we refer to Goodearl [6]. Using the main theorem of [14] we can show that a ring is a right SI domain if and only if it is a right PCI domain. PCI domains are introduced and investigated by Faith in [5].

Corollary 2.7. *For a simple ring R the following conditions are equivalent:*

(i) *Every cyclic singular right R-module is quasi-continuous,*

(ii) *There exists a cyclic right R-module X with $X \neq Z(X)$ such that every cyclic X-singular module in $\sigma[X]$ is quasi-continuous.*

In this case, R is right SI.

Proof. We need to show (ii)⇒(i). With the same argument as that in the proof of Corollary 2.5 we obtain Mod-$R = \sigma[X]$. Hence (ii)⇒(i). Under either (i) or (ii), Theorem 2.5 says that R/E is semisimple for each essential right ideal $E \subseteq R$. Since R is nonsingular, R is right SI by [6, 3.1]. □

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