ACTA MATHEMATICA VIETNAMICA
Volume 31, Number 2, 2006, pp. 131-136

SOLVABLE SUBGROUPS IN THE DIVISION RING OF REAL QUATERNIONS

BUI XUAN HAI AND LE KHAC HUYNH

Abstract. Maximal solvable subgroups of the multiplicative group H^* of the division ring H of real quaternions were described in [2]. In this paper we study the structure of the solvable subgroups of H^*.

1. Introduction

Let H be the division ring of real quaternions. Then the center of H is the field $\mathbb{R}$ of real numbers. If we consider H as the vector space over $\mathbb{R}$, then the set $\{1, i, j, k\}$ is the basis of H. Note that, all other symbols and notations in this paper are standard.

In [1] the authors conjectured that there are no maximal solvable subgroups of the multiplicative group of a division ring, provided it is non-commutative. However, M. Mahdavi-Hezavehi [5] successfuly constructed the solvable maximal subgroup $M_H := \mathbb{C}^* \cup \mathbb{C}^* j$ of the multiplicative group of the division ring of real quaternions H, so he gave a negative answer to the conjecture mentioned above. In [2], we have proved that every solvable maximal subgroup of H^* is conjugate with M_H in H^*. So, all solvable maximal subgroups of H^* are described. In this paper, we are interested in the problem of describing all the solvable subgroups of H^*.

2. Solvable subgroups containing a non-central abelian normal subgroup

Theorem 2.1. *Let S be a solvable subgroup of H^*. If there exists in S a non-central abelian normal subgroup, then either S is abelian or it is contained in some maximal solvable subgroup of H^*.*

Proof. Let N be a non-central abelian normal subgroup of S. Then there exists a non-central element $u \in N$. Clearly, $K := \mathbb{R}(u)$ is a maximal subfield of H. Since N is abelian, $N \subseteq C_H(K) = K$, where $C_H(K)$ denotes the centralizer of K in H. Moreover, since $N \trianglelefteq S$, it follows that $S \subseteq N_{H^*}(K^*)$. For any element $a \in S$, define the map $\Phi_a : K \longrightarrow K$ by $\Phi_a(x) = axa^{-1}, \forall x \in K$. Clearly, $\Phi_a \in Gal(K/\mathbb{R})$. Now, let us consider the group homomorphism $f : S \longrightarrow Gal(K/\mathbb{R})$, defined by $a \mapsto \Phi_a$. Since $Ker f = C_S(K), S/C_S(K) \simeq Im f \leq Gal(K/\mathbb{R})$.

Received September 20, 2005.

Clearly, $[K:\mathbb{R}]=2$ and it follows that K is a Galois extension over $\mathbb{R}$. Therefore $|Gal(K/\mathbb{R})|=[K:\mathbb{R}]=2$. It follows, either $S=C_S(K)$ or $[S:C_S(K)]=2$.

If $S=C_S(K)$, then $S\subseteq C_H(K)$. Since K is a field, $K\subseteq C_H(K)$, so either $C_H(K)=K$ or $C_H(K)=H$. Since $K\neq\mathbb{R}, C_H(K)\neq H$. Hence $C_H(K)=K$. So $S\subseteq K$ and, as a consequence, S is abelian.

Now, suppose that $[S:C_S(K)]=2$. Then, there exists some element $b\in S$ such that $S=C_S(K)\cup C_S(K)b$. We claim that $b^2\in\mathbb{R}$. Since $[K:\mathbb{R}]=2$, we can write $K=\mathbb{R}(w)$ with $w^2=-1$. Then

$$b^2\in C_S(K)=C_S(\mathbb{R}(w))=C_S(w)\subseteq C_H(w)=\mathbb{R}(w).$$

Since w is a root of the minimal polynomial $p(X):=min(\mathbb{R},w)$ of the element w over $\mathbb{R}$ and $\Phi_b\in Gal(K/\mathbb{R}),\Phi_b(w)=bwb^{-1}$ is a root of $p(X)$ too. Hence $bwb^{-1}=-w$ or $bw=-wb$. In particular, $bw\neq wb$ and it follows that $b^2\in\mathbb{R}(b)\cap\mathbb{R}(w)=\mathbb{R}$. Since $b\notin\mathbb{R}, b^2<0$. Therefore, there exists an element $s\in\mathbb{R}$ such that $s^2=-b^2$. By setting $\theta:=bs^{-1}$, we have

$$\theta^2=w^2=-1,\theta w=-w\theta,K^*\cup K^*b=\mathbb{R}(w)^*\cup\mathbb{R}(w)^*\theta.$$

In [2, Proposition 3] it was proven that $\mathbb{R}(w)^*\cup\mathbb{R}(w)^*\theta$ is a solvable maximal subgroup of H^*. Therefore, $S=C_S(K)\cup C_S(K)b$ is contained in a solvable maximal subgroup of H^*. □

Definition 2.1. Suppose that S is a solvable subgroup of H^*. We say that S is a *solvable subgroup of type 1* if it contains an abelian non-central normal subgroup (i.e. if S satisfies the condition in Theorem 2.1). Otherwise, we say that S is a *solvable subgroup of type 2.*

Lemma 2.1. *Non-central subgroup S of H^* is solvable of type 1 if and only if $\mathbb{R}^*S$ is solvable of type 1.*

Proof. Suppose S is a non-central solvable subgroup of type 1. Then, there exists some non-central abelian normal subgroup N of S. Clearly, $\mathbb{R}^*N$ is a non-central abelian normal subgroup of $\mathbb{R}^*S$. Hence $\mathbb{R}^*S$ is a solvable subgroup of type 1.

Conversely, suppose that $\mathbb{R}^*S$ is a non-central solvable subgroup of type 1. Then, there exists some non-central abelian normal subgroup M of $\mathbb{R}^*S$. Clearly, so is $\mathbb{R}^*M$. Put $N:=\mathbb{R}^*M\cap S$. Since $\mathbb{R}^*M$ is non-central, there exists some non-central element $a\in S$ and $\alpha\in\mathbb{R}^*$ such that $\alpha a\in\mathbb{R}^*M$. It follows that $a=\alpha^{-1}(\alpha a)\in\mathbb{R}^*M$. Therefore $a\in\mathbb{R}^*M\cap S=N$. So N is a non-central abelian normal subgroup of S. □

Definition 2.2. We say that a subgroup Q of H^* is a *quaternion subgroup* if there are exist some elements a and b in H^* with $a^2=b^2=-1, ab=-ba$ and $Q=\langle a,b\rangle$ (a subgroup of H^* generated by a and b).

It is easy to check that

$$Q=\{1,a,b,ab,-1,-a,-b,-ab\}.$$

Clearly, Q is a solvable subgroup of type 1.

As an example, we note that the set

$$Q_H := \{1, i, j, k, -1, -i, -j, -k\}$$

is one of quaternion subgroups of H^*.

From the definition it is obvious that if Q is a quaternion subgroup of H^*, then every subgroup of H^* which is conjugate with Q is a quaternion subgroup too. The following result shows that by conjugation we can obtain all quaternion subgroups.

Proposition 2.1. *Every quaternion subgroup of H^* is conjugate with Q_H.*

Proof. Let $Q = \langle a, b\rangle$ be an arbitrary quaternion subgroup of H^*. Consider the $\mathbb{R}$-algebra homomorphism $f : H \longrightarrow H$ which is defined by $f(1) = 1, f(i) = a, f(j) = b, f(k) = ab$. It can be shown that, the set $\{1, a, b, ab\}$ is a basis of H over $\mathbb{R}$. Hence f is an $\mathbb{R}$-automorphism of H. So, by Skolem-Noether Theorem (see, for example, [3, p.39]), f is an inner automorphism. Hence, there exists some element $u \in H^*$ such that $f(x) = uxu^{-1}, \forall x \in H^*$. On the other hand, $f(Q_H) = Q$, so $Q = uQ_Hu^{-1}$. □

Lemma 2.2. *Assume that $a, b \in H$ with $[a, b] := aba^{-1}b^{-1} \in \mathbb{R}$. If a and b don't commute with each other, then $ab = -ba$. Moreover, $a^2, b^2 \in \mathbb{R}$.*

Proof. Let us consider the reduced norm of $H/\mathbb{R}$, denoted by RN. Suppose $aba^{-1}b^{-1} = s \in \mathbb{R}$. By taking the reduced norm, from this equality it follows that $s^2 = 1$. Since $ab \neq ba$, this implies $s = -1$. Hence $ab = -ba$. Now, we have

$$a^2b = a(ab) = a(-ab) = -(ab)a = ba^2.$$

So, $a^2 \in C_H(b) \cap C_H(a) = \mathbb{R}$. Similarly, it can be shown that $b^2 \in \mathbb{R}$. □

Lemma 2.3. *Let G be a non-abelian subgroup of H^*, containing $\mathbb{R}^*$ with $[G, G] \subseteq \mathbb{R}^*$. Then, there exists in G a quaternion subgroup Q_G such that $G = \mathbb{R}^*Q_G$. In particular, G is a solvable subgroup of type 1.*

Proof. Since G is non-abelian, there are exist $a, b \in G$ with $ab \neq ba$. By our assumption, $[a, b] \in \mathbb{R}^*$. Then, in view of Lemma 2.2, $ab = -ba, a^2 \in \mathbb{R}, b^2 \in \mathbb{R}$. Since a, b are both non-central, we can find some $s, t \in \mathbb{R}^*$ such that $a^2 = -s^2$ and $b^2 = -t^2$. By setting $a_0 := as^{-1}, b_0 := bt^{-1}$, we have $a_0, b_0 \in G$ and $a_0^2 = b_0^2 = -1, a_0b_0 = -b_0a_0$. Thus, $Q_G = \langle a_0, b_0\rangle$ is a quaternion subgroup which is contained in G. Now, we show that $G = \mathbb{R}^*Q_G$. Thus, suppose there exists an element $c \in G \setminus \mathbb{R}^*Q_G$. There are the following two cases:

a) $c \in C_H(a_0) \cup C_H(b_0) \cup C_H(a_0b_0)$.

First, suppose $c \in C_H(a_0)$. Clearly, $C_H(a_0) = \mathbb{R}(a_0)$. Then $c = \alpha + \beta a_0$ with $\alpha, \beta \in \mathbb{R}$. Since $c \notin \mathbb{R}, \beta \in \mathbb{R}^*$. On the other hand, since $b_0 \notin C_H(a_0)$, it follows that $b_0c \neq cb_0$. Hence, by Lemma 2.2, $b_0c = -cb_0$. Thus, $b_0(\alpha + \beta a_0) = -(\alpha+\beta a_0)b_0$, and it follows that $2\alpha b_0 = 0$, so $\alpha = 0$. Therefore $c = \beta a_0 \in \mathbb{R}^*Q_G$, that is a contradiction.

Now, if $c \in C_H(b_0)$ or $c \in C_H(a_0b_0)$ then, similarly as above, we can obtain a contradiction.

b) $c \notin C_H(a_0) \cup C_H(b_0) \cup C_H(a_0b_0)$.

Then

$$a_0c = -ca_0, b_0c = -cb_0 \text{ and } (a_0b_0)c = -c(a_0b_0).$$

From the first and second equalities it follows

$$(a_0b_0)c = a_0(b_0c) = -a_0(cb_0) = -(a_0c)b_0 = (ca_0)b_0 = c(a_0b_0).$$

But this is a contradiction with the last equality. Thus, the proof is now completed. □

Theorem 2.2. *Let G be a non-abelian subgroup of H^* with $[G, G] \subseteq \mathbb{R}^*$. Then, G is a solvable subgroup of type 1.*

Proof. Clearly, the subgroup $\mathbb{R}^*G$ satisfies the condition of Lemma 2.3. So, there exists a quaternion subgroup Q such that $\mathbb{R}^*G = \mathbb{R}^*Q$. By Lemma 2.1, $\mathbb{R}^*Q$ is solvable of type 1. Hence, again by Lemma 2.1, G is solvable of type 1. □

Lemma 2.4. *If Q is a quaternion subgroup of H^*, then*

$$Q \subseteq [H^*, H^*] \text{ and } [Q, Q] = \{\pm 1\}.$$

Proof. Suppose $Q = \langle a, b\rangle$ with $a^2 = b^2 = -1, ab = -ba$. Clearly,

$$min(\mathbb{R}, a) = min(\mathbb{R}, b) = min(\mathbb{R}, ab) = X^2 + 1.$$

By Dickson Theorem (see [5, Th.(16.8), p.265]), there exist elements $u, v \in H^*$ such that

$$b = uau^{-1} \text{ and } ab = vav^{-1}.$$

Therefore,
$ab = a(uau^{-1}) = -a^{-1}(uau^{-1}) = -[a^{-1}, u] \in [H^*, H^*]$;
$b = a^{-1}vav^{-1} = [a^{-1}, v] \in [H^*, H^*]$;
$a = (ab)b^{-1} \in [H^*, H^*]$ and
$-1 = aba^{-1}b^{-1} \in [H^*, H^*]$.
Hence $Q \subseteq [H^*, H^*]$.
Direct calculations show that $[Q, Q] = \{\pm 1\}$. □

Theorem 2.3. *Let Q be a non-abelian subgroup of H^*. Then the following statements are equivalent:*
(i) *Q is a quaternion subgroup.*
(ii) *$Q \subseteq [H^*, H^*]$ and $[Q, Q] \subseteq \mathbb{R}^*$.*

Proof. In view of Lemma 2.4, it remains to prove the implication (ii) $\Longrightarrow$ (i). Thus, suppose (ii) holds. Since $[Q, Q] \subseteq \mathbb{R}^*$, by Lemma 2.3 there exists some quaternion subgroup $Q_0 \leq \mathbb{R}^*Q$ such that $\mathbb{R}^*Q = \mathbb{R}^*Q_0$. We now prove that $Q = Q_0$.

For every $x \in H$, denote by $RN(x)$ its reduced norm of H to $\mathbb{R}$. Now, consider $x \in Q$. Then, there exist $\alpha \in \mathbb{R}^*$ and $u \in Q_0$ such that $x = \alpha u$. Note that from Lemma 2.4 it follows that the reduced norm of any element of a quaternion subgroup is 1 . Moreover, since $x \in Q \subseteq [H^*, H^*], RN(x) = 1$. Therefore, by taking the reduced norm, from the equality $x = \alpha u$, we obtain $\alpha^2 = 1$. Hence $\alpha = 1$ or $\alpha = -1$. Therefore $x = u$ or $x = -u$. Recall that $-1 \in Q_0$, hence $x = \pm u \in Q_0$. Thus, $Q \subseteq Q_0$. Since Q is non-abelian, Q must be equal to Q_0. □

Corollary 2.1. *Let Q and Q_0 be quaternion subgroups. If $\mathbb{R}^*Q \subseteq \mathbb{R}^*Q_0$, then $Q = Q_0$.*

Proof. Similarly as in the proof of Theorem 2.3, we see that if $\mathbb{R}^*Q \subseteq \mathbb{R}^*Q_0$, then $Q \subseteq Q_0$. Since Q and Q_0 have the same finite order, it follows $Q = Q_0$. □

3. Solvable subgroups containing no non-central abelian normal subgroups

Theorem 3.1. *Let S be a non-central solvable subgroup of type 2. Then, there exists some positive integer n such that $S^{(n)}$ is a quaternion subgroup.*

Proof. Since S is a non-central solvable subgroup of type 2, S is non-abelian. By Theorem 2.2, $S^{(1)} := [S, S] \nsubseteq \mathbb{R}^*$. Since S is solvable, there exists some integer $n \geq 0$ such that $S^{(n+1)}$ is a non- trivial abelian normal subgroup of S. Since S is a solvable subgroup of type 2, $S^{(n+1)}$ is central. The fact that $S^{(1)}$ is non-central forces $n \geq 1$. Therefore $S^{(n+1)} = [S^{(n)}, S^{(n)}] \subseteq \mathbb{R}^*$. Moreover, since $S^{(n)} \subseteq [H^*, H^*]$, $S^{(n)}$ is a quaternion subgroup by Theorem 2.3. □

Now, we will consider the structure of non-central finite solvable subgroups of type 2. By Theorem 3.1, if A is such a subgroup, then $A^{(n)}$ is a quaternion subgroup for some $n \geq 1$. If $n \geq 2$, then for $S = A^{(n-1)}$ we have $S \subseteq [H^*, H^*]$ and $[S, S]$ is a quaternion subgroup. The following theorem gives some characterization of such subgroups.

Theorem 3.2. *Let S be a finite non-central solvable subgroup of type 2, such that $S \subseteq [H^*, H^*]$ and $[S, S]$ is a quaternion subgroup. Then, one of the following cases occurs:*

(a) *S is a 2-group;*

(b) *$S = QT$, where Q is a quaternion subgroup, which is normal in S, T is a subgroup of odd order of the multiplicative group of some maximal subfield of H;*

(c) *$S = PT$, where P is a normal 2-Sylow subgroup of $S, [P, P] = Q$ is a quaternion subgroup, T is a finite subgroup of odd order of the multiplicative group of some maximal subfield of H.*

Proof. Let S be such a subgroup. Denote by P some 2-Sylow subgroup of S containing $Q := [S, S]$. Then, $k := \dfrac{n}{|P|}$ is a Hall divisor of n. By Hall's Theorem (see, for example [6]) there exists some subgroup T of order k of S. If $T \subseteq \mathbb{R}^*$, then $T \subseteq \{\pm 1\}$. Since T has a odd order, it follows that $T = \{1\}$. So, $S = P$ is a 2-group. Now, suppose that $T \nsubseteq \mathbb{R}^*$. If T is a solvable subgroup of type 2 then, since $T \nsubseteq \mathbb{R}^*$, T is non-abelian. By Theorem 3.1, there exists some integer $r \geq 1$ such that $T^{(r)}$ is a quaternion subgroup. Therefore $\{\pm 1\} \leq T^{(r)} \leq T$ and it follows that the order of T is even, a contradiction. Thus T is a solvable subgroup of type 1. Similarly as in the proof of Theorem 2.1, we can find some maximal subfield K of H such that $|T/C_T(K)| \leq 2$. Since the order of T is odd, $T = C_T(K)$. Again, as in the proof of Theorem 2.1, we can conclude that T is a subgroup of K^*. Since P contains $[S, S]$, P is normal in S. Therefore, PT is a subgroup of S. Moreover, since $(|P|, |T|) = 1$, $P \cap T = \{1\}$. So, it follows that

$PT = S$. Now, consider the subgroup $[P, P] \leq [S, S] = Q$. The order of $[P, P]$ may be one of numbers: $1, 2, 4, 8$. By assumption $Q \leq P$ and Q is a quaternion subgroup (which is non-abelian), so $[P, P] \neq \{1\}$. Suppose that the order of $[P, P]$ is 4. Then, $[P, P]$ is an abelian non-central subgroup of S. Moreover, for every $z \in S$, and $u, v \in P$, since $P \trianglelefteq S$, we have

$$z[u, v]z^{-1} = [zuz^{-1}, zvz^{-1}] \in [P, P].$$

Thus, $[P, P]$ is an abelian non-central normal subgroup of S. But, this contradicts the assumption that S is a solvable subgroup of type 2. Hence, $|[P, P]| = 2$ or $|[P, P]| = 8$. If $[P, P]$ is a subgroup of order 2, then $[P, P] = \{\pm 1\}$. So, by Theorem 2.3, P is a quaternion subgroup, and it follows $P = Q$. If the order of $[P, P]$ is 8, then $[P, P] = Q$. $\square$

References

[1] S. Akbari, M. Mahdavi-Hezavehi, M. G. Mahmudi, *Maximal subgroups of $GL_1(D)$,* J. Algebra **217** (1999), 422-433.

[2] Bui Xuan Hai and Le Khac Huynh, *On maximal subgroups in division rings*, Acta Math. Vietnam. **30**, 1 (2005) 35-43.

[3] P. Draxl, *Skew Fields*, London Mathematical Society, Lecture Note Series Vol. **81** (1983), Cambridge University Press.

[4] T. Y. Lam, *A First Course in Non- Commutative Rings*, GTM No. **131** (1991), Springer-Verlag.

[5] M. Mahdavi - Hezavehi, *Free subgroups in maximal subgroups of $GL_1(D)$*, J. Algebra **241** (2001), 720-730.

[6] J. J. Rotman, *An Introduction to the Theory of Groups*, 4-th Edition, Springer-Verlag, 1995.

Vietnam National University of Ho Chi Minh City
Department of Mathematics & Computer Sciences
227 Nguyen Van Cu street, Dist. 5
Ho Chi Minh City, Vietnam
E-mail address: bxhai@mathdep.hcmuns.edu.vn