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SOME PROPERTIES OF GENERALIZED LOCAL COHOMOLOGY MODULES

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Abstract. Let R be a commutative Noetherian ring, $\mathfrak{a}$ an ideal of R, M and N be two finitely generated R-modules. Let t be a positive integer. We prove that if R is local with maximal ideal $\mathfrak{m}$ and $M \otimes_R N$ is of finite length then $H^t_{\mathfrak{m}}(M, N)$ is of finite length for all $t \geqslant 0$ and

$$l_R(H^t_{\mathfrak{m}}(M, N)) \leqslant \sum_{i=0}^{t} l_R(\operatorname{Ext}^i_R(M, H^{t-i}_{\mathfrak{m}}(N))).$$

This yields $l_R(H^t_{\mathfrak{m}}(M, N)) = l_R(\operatorname{Ext}^t_R(M, N))$.

Additionally, we show that $\operatorname{Ext}^i_R(R/\mathfrak{a}, N)$ is Artinian for all $i \leqslant t$ if and only if $H^i_{\mathfrak{a}}(M, N)$ is Artinian for all $i \leqslant t$. Moreover, we show that whenever $\dim(R/\mathfrak{a}) = 0$ then $H^t_{\mathfrak{a}}(M, N)$ is Artinian for all $t \geqslant 0$.

1. Introduction

Throughout this paper, R is a commutative Noetherian ring with identity and $\mathfrak{a}$ is an ideal of R. For an R-module N and non-negative integer t, the local cohomology module $H^t_{\mathfrak{a}}(N)$ was first introduced and studied by Grothendieck [4]. For example he showed that $H^t_{\mathfrak{m}}(N)$ is Artinian for all t, whenever N is finitely generated and R is local with maximal ideal $\mathfrak{m}$. One of the general problem in local cohomology is to find out when the local cohomology module $H^t_{\mathfrak{a}}(N)$ is Artinian (see [6, Problem 3]). In [7], Melkersson proved that for an R-module N, $H^i_{\mathfrak{a}}(N)$ is Artinian for all $i \leqslant t$ if and only if $\operatorname{Ext}^i_R(R/\mathfrak{a}, N)$ is Artinian for all $i \leqslant t$. The latter result leads us to consider the same result for generalized local cohomology modules which was introduced and studied by Herzog [5] (see also [10]). For each $i \geqslant 0$, the generalized local cohomology functor $H^i_{\mathfrak{a}}(.,.)$ is defined by

$$H^i_{\mathfrak{a}}(M, N) = \varinjlim_{n} \operatorname{Ext}^i_R(M/\mathfrak{a}^n M, N)$$

for all R-modules M and N. Clearly, this is a generalization of the usual local cohomology module. The reader is referred to articles [1], [2] and [3] for more results on generalized local cohomology. Our main purpose in this paper is to show the following.

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Theorem 1.1. *Let $\mathfrak{a}$ be an ideal of R, M a finitely generated R-module, and let t be a positive integer.*

(a) If R is local with maximal ideal $\mathfrak{m}$ and N is a finitely generated R-module such that $M \otimes_R N$ is of finite length, then

$$l_R(H^t_{\mathfrak{m}}(M,N)) \leqslant \sum_{i=0}^{t} l_R(\mathrm{Ext}^i_R(M, H^{t-i}_{\mathfrak{m}}(N))) \quad \textit{for all } t \geqslant 0.$$

(b) Let N be an R- module. Then the following statements are equivalent:
(i) $\mathrm{Ext}^i_R(R/\mathfrak{a}, N)$ *is an Artinian R-module for all $i \leqslant t$.*
(ii) $H^i_{\mathfrak{a}}(M,N)$ *is an Artinian R-module for all $i \leqslant t$.*

Clearly (a) extends the result [9, Theorem 3.2] and (b) is equivalent to the main result of [7, Theorem 1.2].

2. The results

We start this section with the following lemma.

Lemma 2.1. *Let R be a local ring with maximal ideal $\mathfrak{m}$, M and N be two finitely generated R-modules such that $M \otimes_R N$ is of finite length. Then $H^t_{\mathfrak{m}}(M,N)$ is of finite length for all $t \geqslant 0$.*

Proof. Consider the Grothendieck spectral sequence [8, Theorem 11.38]

$$E_2^{p,q} := \mathrm{Ext}^p_R(M, H^q_{\mathfrak{m}}(N)) \underset{p}{\Longrightarrow} H^{p+q}_{\mathfrak{m}}(M,N).$$

for $t \geqslant 0$, we have a finite filtration

$$0 = \phi^{t+1}H^t \subseteq \phi^t H^t \subseteq \cdots \subseteq \phi^1 H^t \subseteq \phi^0 H^t = H^t_{\mathfrak{m}}(M,N)$$

such that $E_\infty^{i,t-i} = \phi^i H^t/\phi^{i+1}H^t$ for all $0 \leqslant i \leqslant t$. Now, since $E_\infty^{i,j}$ is a homomorphic image of $E_2^{i,j}$ for all $i,j \geqslant 0$, by [9, Lemma 3.1], $E_2^{i,j}$ is of finite length for all $i,j \geqslant 0$. It therefore follows that $E_\infty^{i,j}$ is of finite length for all $i,j \geqslant 0$ and that $\phi^{t-i}H^t$ is of finite length for all $0 \leqslant i \leqslant t$. Hence, using the exact sequence

$$0 \longrightarrow \phi^1 H^t \longrightarrow H^t_{\mathfrak{m}}(M,N) \longrightarrow E_\infty^{0,t} \longrightarrow 0 \quad (0 \leqslant i \leqslant t)$$

we get $H^t_{\mathfrak{m}}(M,N)$ is of finite length for all $t \geqslant 0$. □

The following theorem extends [9, Theorem 3.2].

Theorem 2.2. *Let R be a local ring with maximal ideal m and let M and N be two finitely generated R-modules such that $M \otimes_R N$ is of finite length. Then*

$$l_R(H^t_{\mathfrak{m}}(M,N)) \leqslant \sum_{i=0}^{t} l_R(\mathrm{Ext}^i_R(M, H^{t-i}_{\mathfrak{m}}(N)))$$

for all $t \geqslant 0$. Consequently, $l_R(H^t_{\mathfrak{m}}(M,N)) = l_R(\mathrm{Ext}^t_R(M,N))$.

Proof. Let $2 \leqslant i \leqslant t$. With the notation of [8, §11] we consider the exact sequences

$$0 \longrightarrow \ker d_i^{i,t-i} \longrightarrow E_i^{i,t-i} \longrightarrow E_i^{2i,t-2i+1}$$

and note that $E_i^{i,t-i} = \ker d_{i-1}^{i,t-i}/\operatorname{im} d_{i-1}^{1,t-2}$ and $E_i^{i,j} = 0$ for all $j < 0$. So, we have $\ker d_{t+2}^{i,t-i} \cong E_{t+2}^{i,t-i} \cong \ldots \cong E_\infty^{i,t-i}$ for all $(0 \leqslant i \leqslant t)$. Now, using the exact sequences

$$0 \longrightarrow \phi^{i+1}H^t \longrightarrow \phi^i H^t \longrightarrow E_\infty^{i,t-i} \longrightarrow 0 \quad (0 \leqslant i \leqslant t)$$

and an argument similar to that used in 2.1 together with the facts

$$E_\infty^{i,t-i} \cong \ker d_{t+2}^{i,t-i} \subseteq \ker d_2^{i,t-i} \subseteq E_2^{i,t-i} \quad \text{for all } 0 \leqslant i \leqslant t,$$

one can deduce that

$$l_R(H_{\mathfrak{m}}^t(M,N)) \leqslant \sum_{i=0}^{t} l_R(\operatorname{Ext}_R^i(M, H_{\mathfrak{m}}^{t-i}(N))).$$

Moreover, there is a spectral sequence

$$E_2^{i,j} = \operatorname{Ext}_R^i(M, H_{\mathfrak{m}}^j(N)) \underset{i}{\Longrightarrow} \operatorname{Ext}_R^{i+j}(M,N).$$

Hence, we have a finite filtration

$$0 = \psi^{t+1}H^t \subseteq \psi^t H^t \subseteq \cdots \subseteq \psi^1 H^t \subseteq \psi^0 H^t = \operatorname{Ext}_R^t(M,N),$$

such that $E_\infty^{i,t-i} = \psi^i H^t/\psi^{i+1}H^t$ for all $0 \leqslant i \leqslant t$. Now, using the same arguments as above, we get $\psi^i H^t/\psi^{i+1}H^t = \varphi^i H^t/\varphi^{i+1}H^t$ for all $0 \leqslant i \leqslant t$ and so the result follows. □

The following corollary extends [9, Corollary 3.3].

Corollary 2.3. *Let the situation be as in Theorem 2.2. Assume that N is Cohen-Macaulay with $\dim N = d$. Then $H_{\mathfrak{m}}^t(M,N) \cong \operatorname{Ext}_R^t(M,N)$.*

Proof. By the same arguments as in the proof of Theorem 2.2 and [9, Corollary 3.3], we have

$$\operatorname{Ext}_R^t(M,N) \cong \operatorname{Ext}_R^{t-d}(M, H_{\mathfrak{m}}^d(N)) \cong E_\infty^{t-d,d},$$

$$0 = \phi^t H^t = \phi^{t-1}H^t = \ldots = \phi^{t-d+1}H^t$$

and, also,

$$\phi^{t-d}H^t = \ldots = \phi^0 H^t = H_{\mathfrak{m}}^t(M,N)$$

for all t. It therefore follows that $H_{\mathfrak{m}}^t(M,N) \cong \operatorname{Ext}_R^t(M,N)$. □

The following theorem is related to [7, Theorem 1.2].

Theorem 2.4. *Let N be an R-module and t a positive integer. Then the following conditions are equivalent:*

(i) $\operatorname{Ext}_R^i(R/\mathfrak{a}, N)$ *is Artinian for all* $i \leqslant t$.

(ii) $H_{\mathfrak{a}}^i(M,N)$ *is Artinian for any finitely generated R-module M and all* $i \leqslant t$.

Proof. (ii) $\Longrightarrow$ (i) is immediate by [7, Theorem 1.2].

(i) $\Longrightarrow$ (ii). By [7, Theorem 1.2] $H^i_{\mathfrak{a}}(N)$ is Artinian for all $i \leqslant t$. Now, by similar arguments as in the proof of 2.1. One can see that $E^{i,t-i}_{\infty}$ and $\phi^i H^t$ are Artinian for all $0 \leqslant i \leqslant t$. We consider the exact sequences

$$0 \longrightarrow \phi^1 H^i \longrightarrow H^i_{\mathfrak{a}}(M,N) \longrightarrow E^{0,i}_{\infty} \longrightarrow 0 \quad (0 \leqslant i \leqslant t).$$

Hence $H^i_{\mathfrak{a}}(M,N)$ is Artinian for all $i \leqslant t$. □

Corollary 2.5. *Let the situation be as in Theorem 2.4. The following conditions are equivalent:*

(i) $\mathrm{Ext}^t_R(R/\mathfrak{a}, N)$ *is Artinian for all* t.

(ii) $H^t_{\mathfrak{a}}(M,N)$ *is Artinian for all finitely generated* R*-modules* M *and all* t.

The following corollary is a generalization of [3, Theorem 2.2].

Corollary 2.6. *Let the situation be as in Corollary 2.5 and assume that* $\dim(R/\mathfrak{a}) = 0$. *Then* $H^t_{\mathfrak{a}}(M,N)$ *is Artinian for all finitely generated* R*-modules* M *and all* t.

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